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WIMAN'S METHOD AND THE 'FLAT REGIONS' OF INTEGRAL FUNCTIONS

By A. J. MACINTYRE (*Aberdeen*)

[Received 30 October 1937]

1. THE consideration of special integral functions

$$f(z) = \sum_0^\infty c_n z^n \tag{1}$$

whose coefficients are all positive and of simple analytic type, the most familiar being Mittag Leffler's function of order $\rho = 1/\alpha$

$$E_\alpha(z) = \sum_0^\infty \frac{z^n}{\Gamma(1+n\alpha)}, \tag{2}$$

shows that the function is dominated when z is large and positive by a comparatively small group of terms near the greatest one. If $c_N z^N$ is this term, then, as z tends to infinity by positive values,

$$f(z) \simeq \sum_{N-N^{\frac{1}{2}+\epsilon}}^{N+N^{\frac{1}{2}+\epsilon}} c_n z^n. \tag{3}$$

We deduce that $f(z)/z^N$ varies little if z varies from a large positive value so that the change in

$$z^{N^{\frac{1}{2}+\epsilon}}$$

is small. A convenient statement is that

$$f(ze^\tau) = f(z)e^{N\tau}\{1+o(1)\} \tag{4}$$

provided that $\tau = O(N^{-\frac{1}{2}-\epsilon})$.

Results of a similar kind are true for any integral function whether its coefficients are real and regular or not. They were obtained by Wiman from a special case by a somewhat complicated process of comparison of coefficients, and his method has been refined by several subsequent writers. It appears that the statement (4) is true of any integral function for *some* large z at which $|f(z)|$ attains its maximum $M(r)$ for $|z| = r$. There is necessarily a considerable loss of precision, and even the result stated has not been published with $\tau = O(N^{-\frac{1}{2}-\epsilon})$ but only with $\tau = O(N^{-\frac{3}{4}-\epsilon})$.* In so far as the modulus of $f(z)$ is concerned much more precise results have been obtained by entirely different methods since J. M. Whittaker coined the term 'flat region'.† In this connexion the most complete result so far has been

* Valiron (7), p. 101, Theorem 29. † J. M. Whittaker (10).

given by Valiron* and is substantially as follows. If $f(z)$ is an integral function of order ρ, there exist arbitrarily large values of z such that

$$\left|\frac{\log|f(ze^{\tau})|}{\log|f(z)|}-1\right| < \delta \quad \text{for} \quad |\tau| < \frac{\delta\eta(\rho)}{\sqrt{\{\log M(|z|)\}}}, \tag{5}$$

$$\log|f(z)| > h(\rho)\log M(|z|).$$

[$\eta(\rho)$ and $h(\rho)$ are positive but otherwise undetermined.]

The object of the present note is to show that Wiman's argument may be used without any reference to a comparison function, or to the coefficients, for the necessary information may be obtained directly and with greater accuracy by a simple device. The advance is exhibited by our

THEOREM 1. *If the lower order of an integral function $f(z)$ is less than ρ, that is, if*

$$\log M(r) < r^{\rho-\delta}$$

for some arbitrarily large r and fixed positive δ, then there exist arbitrarily large values of z such that

$$f(ze^{\tau}) = f(z)\, e^{N\tau}\{1+\omega(\tau)\}, \tag{6}$$

where $\quad |z| = r, \quad |f(z)| = M(r), \quad N = zf'(z)/f(z) \quad (7)$

and

$$|\omega(\tau)| < \frac{\theta^2(e^{2\rho^2}-1)}{e^{\rho^2}-\theta^2} \tag{8}$$

when

$$|\tau| \leqslant \theta\{\log M(r)\}^{-\frac{1}{2}}. \tag{9}$$

If we discard the information concerning $\arg f(z)$, we obtain

THEOREM 2. *If $\theta < e^{-\frac{1}{2}\rho^2}$, then the conclusion of Theorem 1 includes*

$$|\log|f(ze^{\tau})|-\log M(r)| \leqslant \theta\{\log M(r)+\rho^2\}+\log\frac{e^{\rho^2}-\theta^2}{e^{\rho^2}(1-\theta^2e^{\rho^2})}. \tag{10}$$

2. Proof of Theorems 1 and 2, and Theorem 3

Take $z = re^{i\theta}$ so that $f(z)$ has its maximum modulus $M(r)$. Evidently

$$R\left\{\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right\} = R\left\{ire^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\} = 0. \tag{11}$$

Thus $zf'(z)/f(z)$ is real and clearly lies between the left and right derivatives $d\log M(r)/d\log r$ which exist, the latter being the greater, since $\log M(r)$ is a convex function of $\log r$. Now consider the function

$$\phi(\tau) = \frac{f(ze^{\tau})}{f(z)}e^{-N\tau}, \qquad N = \frac{zf'(z)}{f(z)}, \qquad |f(z)| = M(r). \tag{12}$$

* Valiron (**15**).

We propose to show first that $\phi(\tau)$ is bounded for $|\tau| \leqslant \{\log M(r)\}^{-\frac{1}{2}}$ and then that it is practically constant in a smaller circle for some large r. If we write σ for the real part of τ, then

$$|\phi(\tau)| \leqslant \frac{M(re^{\sigma})}{M(r)} e^{-N\sigma}. \tag{13}$$

Our first step is thus to show that for some large r

$$\log M(re^{\sigma}) - \log M(r) - N\sigma \leqslant A \tag{14}$$

for a fixed A and all σ satisfying $|\sigma| \leqslant \{\log M(r)\}^{-\frac{1}{2}}$. The expression on the left in (14) represents the height of the curve

$$y = \log M(e^{x}) \tag{15}$$

at the point $x = \log r + \sigma$ above the 'tangent'

$$y = \log M(r) + N(x - \log r) \tag{16}$$

which 'touches' (15) at the point $x = \log r$. Since (15) is a convex curve, this height is positive whatever the sign of σ and increases with $|\sigma|$. Hence, if the real part of τ lies between $-\sigma$ and σ,

$$\log|\phi(\tau)| \leqslant \log M(re^{\sigma}) + \log M(re^{-\sigma}) - 2\log M(r), \tag{17}$$

where on the right we have added the extreme values of the height.

We now use a lemma whose proof is postponed, viz.

LEMMA 1. *If*

(i) *$\mu(t)$ is positive, increasing and convex,*

(ii) $\mu(t+\sigma_t) + \mu(t-\sigma_t) - 2\mu(t) > A^2$, for $\sigma_t = \{\mu(t)\}^{-\frac{1}{2}}$, (18)

then

$$\mu(t) > e^{(A-\epsilon)t} \tag{19}$$

for all sufficiently large t.

As a result of this lemma we see that, if the lower order of $f(z)$ is less than ρ, then there exist arbitrarily large values of r for which

$$\log M(re^{\sigma}) + \log M(re^{-\sigma}) - 2\log M(r) < \rho^2, \quad \sigma = \{\log M(r)\}^{-\frac{1}{2}} \tag{20}$$

and hence arbitrarily large values of z for which

$$|\phi(\tau)| < e^{\rho^2} \quad \text{for} \quad |\tau| < \{\log M(r)\}^{-\frac{1}{2}}. \tag{21}$$

Our theorem then follows from a simple application of Schwarz's lemma. For completeness we state and prove our requirements as

LEMMA 2. *If $\phi(\tau)$ is regular for $|\tau| < \sigma$ and satisfies*

$$\phi(0) = 1, \qquad \phi'(0) = 0, \qquad |\phi(\tau)| < M, \tag{22}$$

then for $|\tau| < \theta\sigma$ $(0 < \theta < 1)$ we have

$$|\phi(\tau) - 1| \leqslant \frac{\theta^2(M^2-1)}{M-\theta^2}. \tag{23}$$

The function
$$\frac{M\sigma^2\{\phi(\tau)-1\}}{\tau^2\{\phi(\tau)-M^2\}}$$
is evidently regular and bounded by unity in the circle $|\tau| < \sigma$,* hence, for $|\tau| < \theta\sigma$, we have

$$\begin{aligned} M|\phi(\tau)-1| &\leqslant \theta^2|\phi(\tau)-M^2| \\ &= \theta^2|\phi(\tau)-1+1-M^2| \\ &\leqslant \theta^2|\phi(\tau)-1|+\theta^2(M^2-1), \end{aligned}$$

i.e. $$(M-\theta^2)|\phi(\tau)-1| \leqslant \theta^2(M^2-1),$$

which proves our statement.

Now from (21) our lemma applies to $\phi(\tau)$ as defined by (12) for suitably chosen z with $M = e^{\rho^2}$. Theorem 1 follows at once since

$$\omega(\tau) = \phi(\tau)-1.$$

In order to deduce Theorem 2 we must have an upper bound for N. From (20)
$$\log M(re^{\sigma})-\log M(r) < \log M(r)+\rho^2, \tag{24}$$
and from the convexity property
$$N\sigma < \log M(re^{\sigma})-\log M(r).$$
Hence
$$N\sigma < \{\log M(r)+\rho^2\}. \tag{25}$$
With the notation of Lemma 2 it is clear that, when $|\tau| < \theta\sigma$, $\zeta = \phi(\tau)$ is confined to the circle,
$$M|\zeta-1| < \theta^2|\zeta-M^2|,$$
which meets the real axis in the points
$$\zeta_1 = \frac{M(1-M\theta^2)}{M-\theta^2}, \qquad \zeta_2 = \frac{M(1+M\theta^2)}{M+\theta^2}$$
as ends of a diameter. If $\theta^2 < M^{-1}$, this circle does not contain the origin and, while ζ varies within it,
$$\log \zeta_1 < \log|\zeta| < \log \zeta_2, \qquad |\log|\zeta|| < -\log \zeta_1.$$
Theorem 2 follows immediately from these remarks and (25).

We have confined our attention to functions of finite lower order, but it is clear that the method applies equally well to any other simple system of increasing functions by using a suitable modifica-

* The homographic transformation $\zeta = (w-1)/(w-M^2)$ represents the circle $|w| < M$ on $|\zeta| < 1/M$. $M\{\phi(\tau)-1\}/\{\phi(\tau)-M^2\}$ is bounded by unity in $|\tau| < \sigma$ and has a double zero at $\tau = 0$.

tion of Lemma 1. For integral functions about whose order nothing is known, a relatively simple result is

THEOREM 3. *If $f(z)$ is an integral function and $z = re^{i\theta}$ is the point at which $|f(z)|$ attains its maximum modulus $M(r)$, then*

$$f(ze^{\tau}) = f(z)e^{N\tau}\{1+\omega(\tau)\}, \qquad N = zf'(z)/f(z), \tag{26}$$

where $$|\omega(\tau)| < \{\log M(r)\}^{-\epsilon} \quad \text{for} \quad |\tau| < \{\log M(r)\}^{-\frac{1}{2}-\epsilon} \tag{27}$$

for all sufficiently large r outside a set of intervals in which the total variation of $\log r$ is finite.

This follows from Lemma 2 with

$$\sigma = \{\log M(r)\}^{-\frac{1}{2}-\epsilon/3}, \qquad \theta = \{\log M(r)\}^{-2\epsilon/3}, \qquad M = e^2, \tag{28}$$

and the following lemma applied to $\log M(e^t)$.

LEMMA 3. *If $\mu(t)$ is a positive increasing and convex function of t and ϵ is a fixed positive number, the inequality*

$$\mu(t+\sigma_t)+\mu(t-\sigma_t)-2\mu(t) < 2, \qquad \sigma_t = \{\mu(t)\}^{-\frac{1}{2}-\epsilon} \tag{29}$$

holds for all sufficiently large t outside a set of intervals of finite total length.

3. Proof of Lemmas 1 and 3

The proof of Lemma 1 is due to Professor E. M. Wright and is as follows:

We are given

(i) $\mu(t)$ is positive increasing and convex;

(ii) $\mu(t+\sigma_t)+\mu(t-\sigma_t)-2\mu(t) > A^2$, $\sigma_t = \{\mu(t)\}^{-\frac{1}{2}}$.

Since $\mu(t)$ is convex, it possesses both a right and left derivative for all values of t. We use $\mu'(t)$ to denote the latter. Since $\mu(t)$ and $\mu'(t)$ are both increasing, they are both integrable.

From (i) and (ii) it follows that*

$$\mu(t+\sigma)+\mu(t-\sigma)-2\mu(t) > A^2 \tag{30}$$

for all $\sigma > \sigma_t$, and, since σ_t decreases,

$$\mu(T+\sigma_t)+\mu(T-\sigma_t)-2\mu(T) > A^2$$

for all $T > t$. Hence

$$\mu(t+2\sigma_t)+\mu(t)-2\mu(t+\sigma_t) > A^2,$$
$$\mu(t+3\sigma_t)+\mu(t+\sigma_t)-2\mu(t+2\sigma_t) > A^2,$$
$$\cdots\cdots\cdots$$
$$\mu(t+n\sigma_t)+\mu(t+[n-2]\sigma_t)-2\mu(t+[n-1]\sigma_t) > A^2.$$

* $\mu(t+\sigma)+\mu(t-\sigma)-2\mu(t)$ is an increasing function of σ by (i).

Adding, we have

$$\mu(t+n\sigma_t)-\mu(t+[n-1]\sigma_t) > (n-1)A^2+\mu(t+\sigma_t)-\mu(t). \tag{31}$$

Now

$$\sigma_t\mu'(t+n\sigma_t) \geqslant \mu(t+n\sigma_t)-\mu(t+[n-1]\sigma_t),$$

$$\mu(t+\sigma_t)-\mu(t) \geqslant \sigma_t\mu'(t),$$

so that (31) involves

$$\sigma_t\mu'(t+n\sigma_t) \geqslant (n-1)A^2+\sigma_t\mu'(t),$$

i.e.

$$\mu'(t+n\sigma_t)-\mu'(t) \geqslant (n-1)A^2/\sigma_t. \tag{32}$$

Now let δ be a positive number independent of t. Choose

$$n = [\delta/\sigma_t], \tag{33}$$

and (34) gives

$$\mu'(t+\delta)-\mu'(t) > (n-1)A^2/\sigma_t > \delta(A^2-\epsilon)\mu(t)$$

$$[t \geqslant t_0 = t_0(\epsilon,\delta)]. \tag{34}$$

Integrating (34) from t_0 to t gives

$$\mu(t+\delta)-\mu(t) > \delta(A^2-\epsilon)\int_{t_0}^{t}\mu(t)\,dt. \tag{35}$$

Now suppose $\mu(T) > Ce^{\lambda T}$ for $t_0 \leqslant T \leqslant t$. Then, if

$$K = \delta(A^2-\epsilon)e^{\lambda t_0}/\lambda,$$

$$\mu(t+\delta) > Ce^{\lambda t}+\frac{C\delta(A^2-\epsilon)}{\lambda}e^{\lambda t}-KC$$

$$\geqslant Ce^{\lambda(t+\delta)},$$

provided that

$$e^{\lambda\delta} \leqslant 1+\frac{\delta(A^2-\epsilon)}{\lambda}-Ke^{-\lambda t}.$$

This last is true provided that

$$\lambda \leqslant \sqrt{(A^2-2\epsilon)}, \qquad \delta < \delta_0 = \delta_0(\epsilon), \qquad t \geqslant t_1 = t_1(\epsilon).$$

That is, if $\mu(T) > Ce^{\lambda T}$ for $t_0 < T < t$, it is true for $T = t+\delta$. Hence, for all $t > t_0$ and some positive C $(= C[t_1])$,

$$\mu(t) > Ce^{(A-\epsilon)t}. \tag{36}$$

It may be as well to recapitulate, giving the logical order of choice of the parameters occurring in the proof.

First choose $\epsilon > 0$ (small). Put $\lambda = \sqrt{(A^2-2\epsilon)}$. Now choose $\delta > 0$ so small that

$$e^{\lambda\delta} < 1+\delta(\lambda+\epsilon/2\lambda).$$

The choice of t_0 is explained in the text and depends on ϵ and δ. K then has the value $\delta(\lambda+\epsilon/\lambda)e^{\lambda t_0}$ and t_1 is taken so large that

$$Ke^{-\lambda t_1} < \delta\epsilon/2\lambda.$$

Finally, since $\mu(t)$ is positive and increasing, we can find $C > 0$ so that
$$\mu(t) > Ce^{\lambda t} \quad (t_0 \leqslant t \leqslant t_1).$$

Lemma 3 appears to be more elementary, and Professor Wright's proof can easily be modified to establish it.

If (29) is not satisfied for all sufficiently large t, let t_0 be a value such that $\mu(t_0) > e$, and
$$\mu(t_0+\sigma)+\mu(t_0-\sigma)-2\mu(t_0) > 2, \qquad \sigma = \{\mu(t_0)\}^{-\frac{1}{2}-\epsilon}. \tag{37}$$
Now suppose t_1 to be the smallest value of $t \geqslant t_0+\sigma$ for which (29) is not satisfied and then, since $\sigma > \{\mu(t_1)\}^{-\frac{1}{2}-\epsilon}$,
$$\mu(t_1+\sigma)+\mu(t_1-\sigma)-2\mu(t_1) > 2. \tag{38}$$
Similarly, let t_{k+1} be the least value of $t \geqslant t_k+\sigma$ for which (29) is not satisfied. Thus for $k = 0, 1,..., n$
$$\mu(t_k+\sigma)+\mu(t_k-\sigma)-2\mu(t_k) > 2, \tag{39}$$
and, since $\mu(t)$ is convex and $t_{k+1} \geqslant t_k+\sigma$,
$$\mu(t_{k+1})-\mu(t_{k+1}-\sigma) \geqslant \mu(t_k+\sigma)-\mu(t_k). \tag{40}$$
From (39), with $k+1$ in place of k, and (40)
$$\mu(t_{k+1}+\sigma)-\mu(t_{k+1}) > 2+\mu(t_k+\sigma)-\mu(t_k) \tag{41}$$
and hence
$$\mu(t_k+\sigma)-\mu(t_k) > 2k+\mu(t_0+\sigma)-\mu(t_0), \tag{42}$$
so that
$$\mu(t_{k+1})-\mu(t_k) > 2k+\mu(t_0+\sigma)-\mu(t_0),$$
and, on adding this inequality with $k = 1, 2,..., n-1$ to the preceding inequality with $k = n$,
$$\mu(t_n+\sigma)-\mu(t_1) > n(n+1).$$
Hence
$$\mu(t_n+\sigma) > n^2. \tag{43}$$
Now take
$$n = [\{\mu(t_0)\}^{\frac{1}{2}+\epsilon/2}]+1, \tag{44}$$
and let T_1 be the least value of $t > t_n+\sigma$ for which (29) is not true. Put T_0 for t_0 and then
$$\mu(T_1) > \{\mu(T_0)\}^{1+\epsilon}. \tag{45}$$
Deduce T_2 from T_1 in the same way that T_1 was obtained from T_0 and so on. Evidently
$$\log\mu(T_n) > (1+\epsilon)^n \log\mu(T_0), \tag{46}$$
so that $\mu(T_n)$, and hence T_n, increase indefinitely with n. Thus the

values of $t > T_0$ for which (29) is not true are enclosed in a sequence of intervals of total length less than

$$\sum_0^\infty \frac{\{\mu(T_n)\}^{\frac{1}{2}+\epsilon/2}+2}{\{\mu(T_n)\}^{\frac{1}{2}+\epsilon}}. \tag{47}$$

From (46) this is a convergent series and the lemma is proved.

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6. G. Valiron, *Ann. de l'École Norm.* (1920), 219–53.
7. ——, *Integral Functions* (Toulouse, 1923), 93–120.
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3 deals only with the relation between the maximum modulus and the maximum term. The rest proceed to weaker forms of our Theorem 3 with applications. For a deeper theory of our relation (3) see:

9. N. Wiener and W. T. Martin, *Duke Math. J.* 3 (1937), 213–23.

FLAT REGIONS.

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12, **14**, and **15** also deal with certain meromorphic functions.

NOTE ON A SPECIAL TYPE OF PLANAR TRIODE

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[Received 18 November 1937]

1. Introduction

In a recent paper* on thermionic vacuum tubes various results were obtained applicable to the case of a planar triode having a plane grid of parallel wires of infinite length and small circular section. The anode and cathode were planes parallel to the plane of the grid-wires. In this paper it was shown that a method due to Maxwell yielded results which could be applied to thermionic tubes whose relative dimensions are those of the tubes normally in use. In the procedure suggested by Maxwell† the electrostatic field due to an infinite row of coplanar parallel wires of small circular section was evaluated and the 'plane' cathode and anode were identified with two slightly sinuous equipotential surfaces both at a great distance from the plane of the grid-wires.

We propose here to discuss the distribution of potential in a planar triode having a grid composed of an infinite row of parallel *strips* of small breadth whose planes are inclined to those of the anode and cathode. By means of a conformal transformation (cf. § 2) the complex electrostatic potential due to the freely charged strips will be deduced from that of the simple problem of a charged circular cylinder of infinite length under the influence of line charges. The complex potential thus found will then be modified so as to include a term corresponding to a state of electrification which produces a uniform field normal to the plane of the grid. At a great distance from the grid the equipotentials will be slightly sinuous surfaces and two such surfaces, one on either side of the grid, will be taken as the anode and cathode of the triode. In effect, a method similar to that of Maxwell will be employed. The application of the conformal transformation to the problem of the distribution of potential in, and the magnification factor of, the triode will be found in § 3.

When the strips of the grid are (i) at right angles, and (ii) parallel, to the planes of the anode and cathode, Appleton,‡

* L. Rosenhead and S. D. Daymond, *Proc. Roy. Soc.* 161 (1937), 382–405.
† J. C. Maxwell, *Treatise on Electricity and Magnetism*, 1 (1892), 310.
‡ E. V. Appleton, *Thermionic Vacuum Tubes* (Methuen, 1933), 42.

applying the method due to Maxwell, finds the following expressions for the magnification:

in case (i) $$b_2\Big/\left\{-\frac{d}{2\pi}\operatorname{logsinh}\frac{\pi c}{d}\right\},$$

in case (ii) $$b_2\Big/\left\{-\frac{d}{2\pi}\operatorname{logsin}\frac{\pi c}{d}\right\}.$$

[See §4 for the significance of these symbols.] In actual fact the method employed is only accurate up to the first order in c/d, so that both $\sinh(\pi c/d)$ and $\sin(\pi c/d)$ should be replaced by $\pi c/d$. In the following investigation we shall show that, *whatever* the inclination of the strips of the grid to the direction of the anode and cathode, the magnification factor is $b_2\Big/\left\{-\frac{d}{2\pi}\log\frac{\pi c}{d}\right\}$ correct to first-order terms in c/d.

2. The conformal transformation

Let z and ζ be the complex variables $x+iy$ and $\xi+i\eta$ respectively and k a real positive quantity less than unity. Consider the effect of the transformation

$$\left(-i\frac{2\pi z}{d}+i\pi\right)e^{-i\alpha} = e^{i\alpha}\log\frac{\zeta-k}{\zeta+k}-e^{-i\alpha}\log\frac{\zeta-k^{-1}}{\zeta+k^{-1}}, \tag{1}$$

on the space outside the circle $|\zeta| = 1$. This transformation is a variation of one due to Kawada.*

If ζ lies on this circle and $\arg\zeta = \theta$, we have

$$-i\frac{2\pi}{d}\{(x-\tfrac{1}{2}d)+iy\}e^{-i\alpha} = e^{i\alpha}\log\frac{e^{i\theta}-k}{e^{i\theta}+k}-e^{-i\alpha}\log\frac{e^{i\theta}-k^{-1}}{e^{i\theta}+k^{-1}},$$

and after simplification this gives

$$(x-nd)\sec\alpha = y\operatorname{cosec}\alpha = \frac{d}{2\pi}\left\{\sin\alpha\log\frac{1+2k\cos\theta+k^2}{1-2k\cos\theta+k^2}-2\theta'\cos\alpha\right\}, \tag{2}$$

where
$$(1-2k^2\cos 2\theta+k^4)^{\frac{1}{2}}\cos\theta' = 1-k^2,$$
$$(1-2k^2\cos 2\theta+k^4)^{\frac{1}{2}}\sin\theta' = 2k\sin\theta,$$

and n is any integer or zero. Thus we have

$$y = (x-nd)\tan\alpha, \tag{3}$$

* S. Kawada, *Proc. Third Int. Cong. of App. Mechanics*, 1 (1930), 393–402.

where α may be considered to lie in the range $-\frac{1}{2}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$. Moreover, if $R^2 = (x-nd)^2+y^2$, we have from (2)

$$R = \frac{d}{2\pi}\left(\sin\alpha \log\frac{1+2k\cos\theta+k^2}{1-2k\cos\theta+k^2}-2\theta'\cos\alpha\right).$$

It will be found that $dR/d\theta$ is zero when

$$(1+k^2)\tan\theta+(1-k^2)\cot\alpha = 0,$$

and thus the maximum and minimum values of R occur when $\theta = -\gamma$ and $\theta = \pi-\gamma$ respectively, where

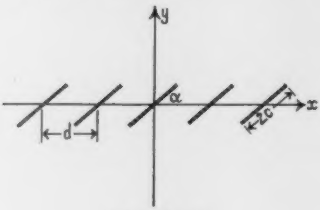
$$(1+k^2)\tan\gamma = (1-k^2)\cot\alpha, \tag{4}$$

and these values are given by

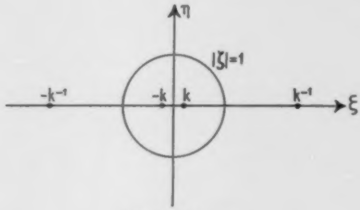
$$R_{\max} = -R_{\min} = \frac{d}{2\pi}\left(\sin\alpha\log\frac{1+2k\cos\gamma+k^2}{1-2k\cos\gamma+k^2}+2\cos\alpha\tan^{-1}\frac{2k\sin\gamma}{1-k^2}\right),$$

where the principal value of the inverse tangent has to be taken.

The (z, ζ) relation expressed in (1) therefore transforms the exterior of the circle $|\zeta| = 1$ in the ζ-plane into the space outside an infinite number of strips of equal length which lie in the parallel straight lines $y = (x-nd)\tan\alpha$ in the z-plane. The length of these straight lines is obviously the difference between the maximum and minimum values of R.



(a) z-plane.



(b) ζ-plane.

FIG 1.

Since $R_{\max}+R_{\min} = 0$, the centres of the strips in the z-plane are spaced at intervals d along the axis of x (Fig. 1 a). Also, if the length of each strip is $2c$, we have, from the last equation and from (4),

$$\begin{aligned} 2c &= R_{\max}-R_{\min} \\ &= \frac{d}{\pi}\left(\sin\alpha\log\frac{1+2k\cos\gamma+k^2}{1-2k\cos\gamma+k^2}+2\cos\alpha\tan^{-1}\frac{2k\sin\gamma}{1-k^2}\right), \end{aligned}$$

i.e.
$$\frac{\pi c}{d} = \sin\alpha\sinh^{-1}\frac{2k\sin\alpha}{1-k^2}+\cos\alpha\sin^{-1}\frac{2k\cos\alpha}{1+k^2}, \tag{5}$$

which determines the constant k in terms of α and $\pi c/d$. As k is less than unity, the functions involved in (5) may be expanded in ascending powers of k, and the equation then becomes

$$\frac{\pi c}{2d} = k-\tfrac{1}{3}k^3\cos 2\alpha+\tfrac{1}{10}k^5(2+\sin^2 2\alpha)+O(k^7).$$

By reversion of this series we get

$$k = \frac{\pi c}{2d}+\frac{1}{3}\left(\frac{\pi c}{2d}\right)^3\cos 2\alpha+O\left\{\left(\frac{\pi c}{2d}\right)^5\right\}. \tag{6}$$

3. The complex potential in the z-plane

It is first necessary to obtain the complex potential appropriate to the problem of the free electrification of the cylinder of unit radius, of which Fig. 1 *b* represents a section normal to the axis of the cylinder. Since the function of ζ expressed in (1) has logarithmic singularities outside the circle $|\zeta| = 1$, at the points $\pm k^{-1}$, the complex potential to be found must also have logarithmic singularities at these points. Obviously, the function $\chi \equiv \phi+i\psi$, where ϕ is the electrostatic potential which arises from line charges, equal to $-\frac{1}{2}e$, $\frac{1}{2}e$ per unit length, passing through the points $\zeta = \pm k^{-1}$, $\zeta = \pm k$ respectively, gives constant potential over the cylinder and also satisfies the condition mentioned above. Thus, if the electrostatic intensity in the direction s is defined to be $-\partial\phi/\partial s$, the required complex potential is given by

$$\chi \equiv \phi+i\psi = -e\log(\zeta^2-k^2)/(1-k^2\zeta^2), \tag{7}$$

the potential ϕ being zero on the cylinder.

The complex potential function χ in the z-plane, arising from the free electrification of the strips, now follows by eliminating ζ from (1) and (7). The elimination of ζ from these equations is quite simple in the two special cases for which α is zero and $\frac{1}{2}\pi$.

In the former case ($\alpha = 0$) we have, from (5),

$$k = \tan(\pi c/2d),$$

and, from (1) and (7),

$$\zeta-\zeta^{-1} = i(k-k^{-1})\tan\frac{\pi z}{d},$$

and
$$\zeta^2+\zeta^{-2}-2 = \frac{(1-k^2)^2(t-1)^2}{(1+k^2t)(k^2+t)},$$

where t is written for $\exp(-\chi/e)$. Hence we have from the last three relations

$$\sin\frac{\pi z}{d} = -i\sin\frac{\pi c}{d}\sinh\frac{\chi}{2e}.$$

Similarly in the latter case ($\alpha = \frac{1}{2}\pi$) we find

$$k = \tanh\frac{\pi c}{2d},$$

and the corresponding equation for the complex potential is

$$\sin\frac{\pi z}{d} = i\sinh\frac{\pi c}{d}\cosh\frac{\chi}{2e}.$$

In both these cases expressions for the complex potentials can be found by more direct methods and they agree with those given above. It will be noted that in the cases dealt with here each strip carries a charge e per unit length, which is therefore the charge of the cylinder in the ζ-plane.

4. Application to the triode

Let us now assume that the grid consists of an infinite series of parallel strips (Fig. 1 a) each of width $2c$ whose centres are spaced at intervals d along the axis of x, and that the cathode and anode coincide with the planes $y = -b_1$, $y = b_2$ respectively. In practical cases the dimensions are such that $(b_1+b_2)/d$ and b_1/d are approximately equal to 5 and 2 respectively, and c/d is not greater than $\frac{1}{10}$. With reference to (6) it will be observed that when c/d is equal to $\frac{1}{10}$, the greatest value yet used in a thermionic tube, the quantity $(k-\pi c/2d)/(\pi c/2d)$ is of the order $\frac{1}{3}(\pi c/2d)^2$, or less than $\frac{1}{100}$. Thus, whatever be the inclination of the strips, the value of k in practical cases may be taken to be $\pi c/2d$.

If the grid is placed in a uniform field E directed towards the cathode, the complex electrostatic potential is of the form

$$\chi \equiv \phi+i\psi = -iEz+\chi'-e\log\frac{\zeta^2-k^2}{1-k^2\zeta^2},$$

where χ' is harmonic in the space outside the grid and must be chosen so as to make ϕ constant on the grid itself.*

* In connexion with this point we are greatly indebted to the referees for giving us the exact form of the complex potential to replace an approximate form which was discussed in the original version of this note. Both the exact and the approximate forms yielded the same result because they were identical to the first order of the small quantities involved and because the following investigation has no pretensions to more than first-order accuracy.

Using (1), we have

$$\chi = \frac{Ed}{2\pi}\left\{-i\pi+e^{2i\alpha}\log\frac{\zeta-k}{\zeta+k}-\log\frac{\zeta-k^{-1}}{\zeta+k^{-1}}\right\}+\chi'-e\log\frac{\zeta^2-k^2}{1-k^2\zeta^2}.$$

Obviously, the term $\quad -i\pi+e^{2i\alpha}\log\frac{\zeta-k}{\zeta+k}$

is harmonic in the grid space and may be absorbed in the function χ'. Thus we now have to determine χ' satisfying the two conditions mentioned above, such that

$$\chi = \frac{Ed}{2\pi}\log\frac{\zeta+k^{-1}}{\zeta-k^{-1}}+\chi'-e\log\frac{\zeta^2-k^2}{1-k^2\zeta^2}.$$

Both conditions are satisfied if we take $\chi' = \frac{Ed}{2\pi}\log\frac{\zeta-k}{\zeta+k}$, so that finally we get

$$\chi \equiv \phi+i\psi = \frac{Ed}{2\pi}\log\left(\frac{\zeta+k^{-1}}{\zeta-k^{-1}}\frac{\zeta-k}{\zeta+k}\right)-e\log\frac{\zeta^2-k^2}{1-k^2\zeta^2}, \tag{8}$$

where ϕ now has the value zero on the strips, and ζ is a function of z given by (1).

Let us now consider the effect of the transformation (1) on the small circle $\zeta = k^{-1}+r\exp(i\omega)$, where for the moment we shall assume that r is at most of the order k. Substituting this value of ζ in (1), we get

$$-i\frac{2\pi}{d}(x+iy) = -\log\tfrac{1}{2}kr-2k^2e^{2i\alpha}-i(\pi+\omega)+\tfrac{1}{2}kre^{i\omega}+O(k^4),$$

and, provided that k^2 is small compared with $-\log\frac{1}{2}kr$, we have

$$\frac{2\pi x}{d} = \pi+\omega, \qquad \frac{2\pi y}{d} = -\log\tfrac{1}{2}kr.$$

Thus the corresponding contour in the z-plane is approximately the straight line $y = b_2$, if

$$\tfrac{1}{2}kr = \exp(-2\pi b_2/d). \tag{9}$$

Now $\exp(-2\pi b_2/d)$ is of the order $\exp(-6\pi)$, while k $(= \pi c/2d)$ is of the order $\frac{1}{6}$, so that both k^2 and kr are very small compared with $-\log\frac{1}{2}kr$, and consequently the above contour is practically a straight line parallel to the axis of x. Moreover, the value ϕ_a of the potential at any point of this approximate straight line (which is the section of the anode made by the z-plane) is easily obtained from (1) and (11) by substituting $k^{-1}+r\exp(i\omega)$ for ζ. We have

$$\phi_a = -\frac{Ed}{2\pi}\log\tfrac{1}{2}kr+e\log 2rk^3,$$

approximately, terms of the order k^2 and kr having been neglected. Substituting $\pi c/2d$ for k and using (9), we get

$$\phi_a = Eb_2 - \frac{2\pi e}{d}\left(b_2 - \frac{d}{\pi}\log\frac{\pi c}{d}\right). \tag{10}$$

In a similar way we find that, to the same degree of accuracy, the small circle $\zeta = -k^{-1}+r'\exp(i\omega')$ in the ζ-plane is transformed into the straight line $y = -b_1$ in the z-plane, where

$$\tfrac{1}{2}kr' = \exp(-2\pi b_1/d), \tag{11}$$

and also the potential ϕ_c at all points on this straight line (which is the section of the cathode) is given by

$$-\phi_c = Eb_1 + \frac{2\pi e}{d}\left(b_1 - \frac{d}{\pi}\log\frac{\pi c}{d}\right). \tag{12}$$

If the differences of potential between anode and grid and between grid and cathode are respectively $v_a - v_g$ and v_g, we have, since in the foregoing discussion the grid is at zero potential,

$$\phi_a = v_a - v_g \quad \text{and} \quad -\phi_c = v_g.$$

We then get, from (10) and (12),

$$v_a = Eb - \frac{2\pi e}{d}(b_2 - b_1), \tag{13}$$

$$v_g = Eb_1 + \frac{2\pi e}{d}b_1 - 2e\log\frac{\pi c}{d}, \tag{14}$$

where $b = b_1 + b_2$.

Now we have, from (1) and (8),

$$\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} = \frac{d\chi}{dz} = \frac{d\chi}{d\zeta}\Big/\frac{dz}{d\zeta} = -i\frac{(1-k^2)}{kd}\,\frac{kdE(1+\zeta^2)-2\pi e\zeta(1+k^2)}{e^{2i\alpha}(1-k^2\zeta^2)+\zeta^2-k^2}.$$

At a point of the cathode we have $\zeta = -k^{-1}+r'e^{i\omega'}$, and to the present degree of accuracy we therefore have, on the cathode,

$$\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} = -i\left(E + \frac{2\pi e}{d}\right),$$

and the surface density of charge there is given by

$$-4\pi\sigma = -\left(\frac{\partial\phi}{\partial y}\right)_{y=-b_1} = E + \frac{2\pi e}{d}. \tag{15}$$

Eliminating E and e from (13), (14), (15), we have

$$-4\pi\sigma = \frac{b_2 v_g + \lambda v_a}{b\lambda + b_1 b_2}, \tag{16}$$

where $2\pi\lambda = -d\log(\pi c/d)$.

The capacity coefficients C_{12}, C_{13} of the grid and anode respectively are obtained by identifying (16) with the equation

$$-4\pi\sigma = 4\pi(C_{12}v_g + C_{13}v_a),$$

and thus

$$4\pi C_{12} = b_2/(b\lambda + b_1 b_2), \qquad 4\pi C_{13} = \lambda/(b\lambda + b_1 b_2).$$

The magnification factor is therefore given by

$$m = C_{12}/C_{13} = b_2/\lambda = b_2 \Big/ \left\{-\frac{d}{2\pi}\log\frac{\pi c}{d}\right\},$$

which is independent of α. We therefore see that, if the grid is composed of strips of small width, the magnification factor is independent of the inclinations of these strips to the directions of the anode and cathode.

ON $\zeta(s)$ AND $\pi(x)$

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1. IN a recent series of papers Vinogradoff and Tchudakoff have obtained important new results on the order of $\zeta(s)$ in the neighbourhood of $\sigma = 1$, and on the order of the remainder in the asymptotic formula for $\pi(x)$, the number of primes up to x. Tchudakoff's papers (**3**)–(**6**) hitherto published depend on Vinogradoff (**1**) (see the list of references at the end). Still better results follow from Vinogradoff (**2**), and these are obtained in the present paper. My main object, however, in writing the paper is to present this rather complicated argument in as simple a form as possible.

2. The chief purpose of the method is to obtain an upper bound for the sum

$$S = \sum_{m=1}^{P} e^{2\pi i f(m)},$$

where

$$f(x) = ax^{n+1}+a_0 x^n+...+a_n$$

is a real polynomial of degree $n+1$.

Let $p_1,..., p_k$ be positive integers to be determined later. Let

$$S_1 = \sum_{x=1}^{p_1} e^{2\pi i f(x+y)}$$

and

$$S_m = \sum_{\mu_m=1}^{p_m} e^{2\pi i f(\nu_2 p_2+...+\nu_m p_m+\mu_m+y)} \quad (m = 2, 3,..., k).$$

Then

$$\sum_{y=1}^{P} \sum_{x=1}^{p_1} e^{2\pi i f(x+y)} = p_1 \sum_{m=p_1}^{P} e^{2\pi i f(m)}+\Big(\sum_{m=2}^{p_1-1}+\sum_{m=P}^{P+p_1}\Big)O(p_1) = p_1 S+O(p_1^2),$$

i.e.

$$S = \frac{1}{p_1} \sum_{y=1}^{P} S_1+O(p_1),$$

the constants of the O's being absolute. Next

$$\sum_{\nu_2=0}^{q_2-1} S_2 = \sum_{\nu_2=0}^{q_2-1} \sum_{\mu_2=1}^{p_2} e^{2\pi i f(\nu_2 p_2+\mu_2+y)} = \sum_{x=1}^{p_2 q_2} e^{2\pi i f(x+y)} = S_1$$

if $p_2 q_2 = p_1$; and so on generally. Hence, if b is another positive integer ($b > n$),

$$|S_1|^{2bk} = |S_1|^{2b}\Big|\sum_{\nu_2=0}^{q_2-1} S_2\Big|^{2b(k-1)} \leqslant |S_1|^{2b} q_2^{2b(k-1)-1} \sum_{\nu_2=0}^{q_2-1} |S_2|^{2b(k-1)},$$

$$|S_2|^{2b(k-1)} \leqslant |S_2|^{2b}\Big|\sum_{\nu_3=0}^{q_3-1} S_3\Big|^{2b(k-2)} \leqslant |S_2|^{2b} q_3^{2b(k-2)-1} \sum_{\nu_3=0}^{q_3-1} |S_3|^{2b(k-2)},$$

and so on, until

$$|S_{k-1}|^{2b.2} \leqslant |S_{k-1}|^{2b} q_k^{2b-1} \sum_{\nu_k=0}^{q_k-1} |S_k|^{2b}.$$

Hence
$$|S_1|^{2bk} \leqslant q_2^{2b(k-1)-1} q_3^{2b(k-2)-1} \dots q_k^{2b-1} \sum_{\nu_2} \dots \sum_{\nu_k} |S_1 \dots S_k|^{2b},$$

and

$$|S| \leqslant p_1^{-1} P^{1-1/2bk} \Big(\sum_{y=1}^{P} |S_1|^{2bk}\Big)^{1/2bk} + O(p_1)$$

$$\leqslant p_1^{-1} P^{1-1/2bk} \Big(q_2^{2b(k-1)-1} \dots q_k^{2b-1} \sum_{\nu_2} \dots \sum_{\nu_k} \sum_{y=1}^{P} |S_1 \dots S_k|^{2b}\Big)^{1/2bk} + O(p_1)$$

$$\leqslant p_1^{-1} P^{1-1/2bk} \Big(q_2^{2b(k-1)} \dots q_k^{2b} \max \sum_{y=1}^{P} |S_1 \dots S_k|^{2b}\Big)^{1/2bk} + O(p_1)$$

$$= P^{1-1/2bk} (p_1 \dots p_k)^{-1/k} \Big(\max \sum_{y=1}^{P} |S_1 \dots S_k|^{2b}\Big)^{1/2bk} + O(p_1). \qquad (2.1)$$

3. Every S_m contains the term $e^{2\pi i f(x_0+y)}$, where

$$x_0 = \nu_2 p_2 + \dots + \nu_k p_k + 1,$$

provided that μ_m takes the value $\nu_{m+1} p_{m+1} + \dots + \nu_k p_k + 1$, which is true, since

$$\nu_{m+1} p_{m+1} + \dots + \nu_k p_k + 1 \leqslant (q_{m+1} - 1) p_{m+1} + \dots + (q_k - 1) p_k + 1$$
$$= (p_m - p_{m+1}) + \dots + (p_{k-1} - p_k) + 1 = p_m - p_k + 1 \leqslant p_m.$$

Hence we may write

$$S_m = \sum_{v_m=-r_m}^{p_m-r_m-1} e^{2\pi i f(x_0+y+v_m)} = \sum_{h=1}^{H} \sum_{v_{m,h}=-r_m+(h-1)p_m/H}^{-r_m+hp_m/H} e^{2\pi i f(x_0+y+v_{m,h})}$$
$$= \sum_{h=1}^{H} S_{m,h},$$

say, where $|r_m| \leqslant p_m$, and H is an integer dividing p_k (and so $p_{k-1}, \dots, p_1$). Hence

$$S_m^b = \sum S_{m,h_1} \dots S_{m,h_b} = \sum C_{m,h_1,\dots,h_b},$$

where $h_1, \dots, h_b$ are chosen independently from the numbers 1 to H. It is then clear, e.g. from the multinomial theorem, that

$$|S_m|^b \leqslant b! \textstyle\sum_1 |S_{m,h_1} \dots S_{m,h_b}| = b! \textstyle\sum_1 |C_{m,h_1,\dots,h_b}|,$$

where, in $\sum_1$, $1 \leqslant h_1 \leqslant h_2 \dots \leqslant h_b \leqslant H$. Thus $\sum_1$ contains not more than H^b terms. Hence

$$|S_1 \dots S_k|^b \leqslant (b!)^k \textstyle\sum_2 |C_1 \dots C_k|,$$

where the first factor is taken from one of the terms of $\sum_1 |C_1|$, etc., C_m being short for

$$C_{m,h_{1,m},\dots,h_{b,m}}.$$

Thus $\sum_2$ contains not more than H^{bk} terms. Hence

$$|S_1 \dots S_k|^{2b} \leqslant (b!)^{2k} \sum_2 1 \sum_2 |C_1 \dots C_k|^2$$
$$\leqslant (b!)^{2k} H^{bk} \sum_2 |C_1 \dots C_k|^2. \tag{3.1}$$

4. We may say that the numbers $h_{1,m}, \dots, h_{b,m}$ are well spaced if there are n of them (at least), say $j_1,\dots, j_n$, none of which lies in the interval $(Hr_m/p_m - \lambda H^{1-1/n}, Hr_m/p_m + \lambda H^{1-1/n}+1)$, and such that

$$j_{\nu+1} - j_\nu \geqslant \lambda H^{1-1/n}+1 \quad (\nu = 1,\dots, n-1), \tag{4.1}$$

where $\lambda = \lambda(n) > 1$ is to be fixed later. Let

$$\sum_2 |C_1 \dots C_k|^2 = \sum_{2,1} + \sum_{2,2},$$

where $\sum_{2,1}$ contains terms in which $C_1,\dots, C_k$ are all formed with well-spaced h's; $\sum_{2,2}$ contains terms in which at least one of $C_1,\dots, C_k$ is not so formed.

The numbers $h_1,\dots$ are certainly well spaced if at least $n+1$ of $h_2-h_1,\dots, h_b-h_{b-1}$ are not less than $2\lambda H^{1-1/n}+1$. Hence, if they are not well spaced, at least $b-n-1$ of these numbers are less than $2\lambda H^{1-1/n}+1$. The number of such sets of h's is less than

$$\frac{b(b-1)\dots(n+1)}{1.2\dots(b-n)} (2\lambda H^{1-1/n}+1)^{b-n-1} H^{n+1} \leqslant b!\,(3\lambda)^{b-n} H^{b-(b-n-1)/n}.$$

Hence the number of terms in $\sum_{2,2}$ does not exceed

$$k.b!\,(3\lambda)^{b-n} H^{b-(b-n-1)/n}.H^{b(k-1)}.$$

Also $$|S_{m,h}| \leqslant p_m/H, \qquad |C_m| \leqslant (p_m/H)^b.$$

Hence

$$\sum_{2,2} |C_1 \dots C_k|^2 \leqslant k.b!\,(3\lambda)^{b-n} H^{-bk-(b-n-1)/n} (p_1 \dots p_k)^{2b}. \tag{4.2}$$

5. In $\sum_{2,1}$ we write

$$C_m = \sum_3 \exp 2\pi i\{f(x_0+y+v_{m,h_{1,m}})+\dots+f(x_0+y+v_{m,h_{b,m}})\},$$

each v going over its proper range, so that $\sum_3$ contains $(p^m/H)^b$ terms. Now

$$f(x_0+y+v_{m,h_{1,m}})+\dots$$
$$= a\{(x_0+y+v_{m,h_{1,m}})^{n+1}+\dots\}+a_0\{(x_0+y+v_{m,h_{1,m}})^n+\dots\}+\dots$$
$$= aV_{m,n+1}+\{a_0+(n+1)a(x_0+y)\}V_{m,n}+\dots,$$

where $$V_{m,r} = v^r_{m,h_{1,m}}+\dots+v^r_{m,h_{b,m}}.$$

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Let
$$U_r = V_{1,r}+\ldots+V_{k,r}.$$
Then
$$C_1 \ldots C_k = \textstyle\sum_4 \exp 2\pi i\{a\,U_{n+1}+\{a_0+(n+1)a(x_0+y)\}U_n+\ldots\},$$
$\sum_4$ containing $(p_1 \ldots p_k)^b H^{-bk}$ terms; and, with an obvious notation,
$$|C_1 \ldots C_k|^2 = \textstyle\sum_4 \sum_4' \exp 2\pi i\{a(U_{n+1}-U'_{n+1})+$$
$$+\{a_0+(n+1)a(x_0+y)\}(U_n-U'_n)+\ldots\}.$$
Hence
$$\sum_{y=1}^{P} \textstyle\sum_{2,1} |C_1 \ldots C_k|^2$$
$$\leqslant \textstyle\sum_{2,1} \sum_4 \sum_4' \left|\displaystyle\sum_{y=1}^{P} \exp 2\pi i\{[a_0+(n+1)a(x_0+y)](U_n-U'_n)+\ldots\}\right|. \quad (5.1)$$

Let the number of solutions of
$$U_n-U'_n = z_n, \quad \ldots, \quad U_1-U'_1 = z_1$$
in this sum, for given $z_1,\ldots, z_n$, be $K(z_1,\ldots,z_n)$. Then the right-hand side is
$$\sum_{z_1,\ldots,z_n} K(z_1,\ldots, z_n)\left|\sum_{y=1}^{P} \exp 2\pi i\{[a_0+(n+1)a(x_0+y)]z_n+\ldots\}\right|$$
$$\leqslant \max K \sum_{z_1,\ldots,z_n}\left|\sum_{y=1}^{P}\right|$$
$$\leqslant \max K\Big(\sum_{z_1,\ldots,z_n} 1\Big)^{\frac{1}{2}}\Big(\sum_{z_1,\ldots,z_n} \sum_{y=1}^{P} \sum_{y'=1}^{P} \exp 2\pi i\{(n+1)a(y-y')z_n+\ldots\}\Big)^{\frac{1}{2}}.$$
Since $|v_{m,h_1}| \leqslant p_1$, etc., $|U_r| \leqslant bkp_1^r$, and z_r goes over not more than $2bk\,p_1^r$ values. Hence
$$\sum_{z_1,\ldots,z_n} 1 \leqslant (2bk)^n p_1^{\frac{1}{2}n(n+1)}.$$
Hence
$$\sum_{y=1}^{P} \textstyle\sum_{2,1}|C_1 \ldots C_k|^2 \leqslant \max K(2bk)^{\frac{1}{2}n} p_1^{\frac{1}{4}n(n+1)}\times$$
$$\times\Big(\sum_{z_1,\ldots,z_{n-1}} \sum_{y} \sum_{y'} \Big|\sum_{z_n} e^{2\pi i(n+1)a(y-y')z_n}\Big|\Big)^{\frac{1}{2}}$$
$$\leqslant \max K(2bk)^{\frac{1}{2}n} p_1^{\frac{1}{4}n(n+1)}\times$$
$$\times\left\{(2bk)^{n-1} p_1^{\frac{1}{2}n(n-1)} \sum_{y=1}^{P} \sum_{y'=1}^{P} \min\left(2bkp_1^n, \frac{1}{|\sin \pi(n+1)a(y-y')|}\right)\right\}^{\frac{1}{2}}.$$
Let $(n+1)|a|P \leqslant \frac{1}{2}$. Then
$$\frac{1}{|\sin \pi(n+1)a(y-y')|} \leqslant \frac{\frac{1}{2}\pi}{\pi(n+1)|a|\,|y-y'|} \leqslant \frac{1}{2|a|\,|y-y'|}.$$

Hence the y, y' sum does not exceed

$$2bkp_1^n P+\frac{P}{|a|}\left(1+\frac{1}{2}+\dots+\frac{1}{P-1}\right) < \left(2bkp_1^n+\frac{2\log P}{|a|}\right)P$$
$$< 4bk|a|^{-1}P\log P$$

if $p_1^n \leqslant 1/|a|$. Hence

$$\sum_{y=1}^{P}\sum\nolimits_{2,1}|C_1\dots C_k|^2 \leqslant 2\max K(2bk)^n p_1^{\frac{1}{2}n^2}|a|^{-\frac{1}{2}}P^{\frac{1}{2}}\log^{\frac{1}{2}} P.$$

6. To obtain an upper bound for $K(z_1,\dots,z_n)$ we use the following lemma.

LEMMA. *Let* $-p \leqslant v_1 < v_2 \dots < v_n \leqslant p$, *and*

$$v_{i+1}-v_i \geqslant pH^{-1/n} \quad (i = 1,2,\dots,n-1).$$

Let $x_1,\dots, x_n$ *satisfy the equations*

$$(1+f_{1,r})v_1^{r-1}x_1+\dots+(1+f_{n,r})v_n^{r-1}x_n = c_r \quad (r = 1,2,\dots,n),$$

where $\quad |c_r| \leqslant cp^{r-1}, \qquad |f_{q,r}| \leqslant 8^{-n}H^{-1+1/n}.$

Then for each r

$$|x_r| \leqslant 2c(4p)^{n-1}/(v_r-v_1)\dots(v_r-v_{r-1})(v_{r+1}-v_r)\dots(v_n-v_r).$$

It is sufficient to consider x_1. Now the result of eliminating x_n between the equations with r and $r+1$ may be written

$$(1+g_{1,r})v_1^{r-1}y_1+\dots = d_r,$$

where $\quad y_1 = (v_n-v_1)x_1, \quad \dots, \quad d_r = \frac{c_r v_n}{1+f_{n,r}}-\frac{c_{r+1}}{1+f_{n,r+1}},$

and $\quad g_{1,r} = \frac{1}{v_n-v_1}\left(\frac{f_{1,r}-f_{n,r}}{1+f_{n,r}}v_n-\frac{f_{1,r+1}-f_{n,r+1}}{1+f_{n,r+1}}v_1\right),\dots.$

Since $|f_{q,r}| \leqslant \frac{1}{2}$, we have

$$|d_r| \leqslant 2p\,.\,cp^{r-1}+2cp^r = 4cp^r$$

and $\quad |g_{1,r}| \leqslant p^{-1}H^{1/n}\,.\,8p\max|f_{q,r}| \leqslant 8^{-n+1}H^{-1+2/n}.$

Repeating the process, we arrive finally at the equation

$$(1+l_{1,1})z_1 = m_1,$$

where

$$z_1 = (v_n-v_1)\dots(v_{n-1}-v_1)x_1,$$
$$|m_1| \leqslant (4p)^{n-1}c, \qquad |l_{1,1}| \leqslant 8^{-1}H^{-1+n/n} = \tfrac{1}{8}.$$

The result now follows.

Since $\quad (v_r-v_1)\dots(v_n-v_r) \geqslant (pH^{-1/n})^{n-1}[\frac{1}{2}n-\frac{1}{2}]!$

we also have $\quad |x_r| < AcH^{1-1/n}, \qquad (6.1)$

where A is an absolute constant.

7. In $\sum_{2,1}$ we have

$$V_{m,r} = v^r_{m,j_{1,m}}+\ldots+v^r_{m,j_{n,m}}+V^{(2)}_{m,r} = V^{(1)}_{m,r}+V^{(2)}_{m,r},$$

where the j's satisfy (4.1). Let N be the number of solutions of

$$V^{(1)}_{1,r}+\ldots+V^{(1)}_{k,r} = W_r \quad (r = 1,\ldots,n)$$

for given $W_1,\ldots,W_n$. Let N_1 be the number of possible sets of $v_{1,j_{1,1}},\ldots, v_{1,j_{n,1}}$ consistent with these equations. Let N_2 be the number of possible sets of $v_{2,j_{1,2}},\ldots, v_{2,j_{n,2}}$ for given W_r and given $v_{1,j_{1,1}},\ldots, v_{1,j_{n,1}}$. Let N_3 be the number of possible sets of $v_{3,j_{1,3}},\ldots, v_{3,j_{n,3}}$ for given W_r, $v_{1,j_{1,1}},\ldots, v_{2,j_{1,2}},\ldots$; and so on. Then

$$N = N_1 N_2 \ldots N_k.$$

Now all values of $V^{(1)}_{m,r}$ lie in an interval of length np^r_m. Hence, for given W_r, $V^{(1)}_{1,r}$ lies in an interval of length

$$n(p^r_2+\ldots+p^r_k) < nkp^r_2.$$

For given W_r, $v_{1,j_1},\ldots, v_{1,j_n}$, i.e. given W_r and $V^{(1)}_{1,r}$, $V^{(1)}_{2,r}$ lies in an interval of length

$$n(p^r_3+\ldots+p^r_k) < nkp^r_3,$$

and so on. Finally, $\sum v^r_{n,j_{1,n}}$ is given for $r = 1,\ldots,n$, so that $v_{n,j_1}\ldots$ are the roots of a given equation of degree n. Hence $N_k \leqslant n^n$.

Consider now N_1. Divide the interval in which $V^{(1)}_{1,r}$ can lie into parts of length not exceeding p_1^{r-1}, the number of such parts being not greater than $[nkp^r_2/p_1^{r-1}]+1$. Let $v_1,\ldots, v_n$ and $v_1+x_1,\ldots, v_n+x_n$ be two sets of values of $v_{1,j_1},\ldots, v_{1,j_n}$ for which $V^{(1)}_{1,r}$ lies in the same partial interval. Then

$$(v_1+x_1)^r-v_1^r+\ldots+(v_n+x_n)^r-v_n^r = c_r, \qquad |c_r| \leqslant p_1^{r-1},$$

for $r = 1,\ldots, n$. Now

$$|(v+x)^r-v^r-rxv^{r-1}| = r(r-1)\left|\int\limits_v^{v+x} d\xi \int\limits_v^{\xi} \eta^{r-2}\,d\eta\right| \leqslant \tfrac{1}{2}x^2r(r-1)(2|v|)^{r-2}$$

if $|x| \leqslant |v|$. Hence

$$(v_1+x_1)^r-v_1^r = rx_1 v_1^{r-1}(1+f_{1,r}),$$

where
$$|f_{1,r}| \leqslant 2^{r-3}(r-1)|x_1/v_1|.$$

Here $|x_1|$, as a difference between two values of v_{m,j_1}, does not exceed p_m/H; and

$$|v_{m,j_1}| \geqslant p_m H^{-1/n}2^{n-3}(n-1)8^n$$

if $\lambda = 2^{4n-3}(n-1)$ in § 4. The conditions of the lemma are therefore satisfied, with $p = p_m$, $c = 1$. Hence

$$|x_1| < AH^{1-1/n} < AH,$$

and similarly for $x_2,\ldots,x_n$. Hence the number of $v_{1,j_1},\ldots, v_{1,j_n}$ for which the $V_{1,r}^{(1)}$ lie in given intervals of length p_1^{r-1} is less than $(2AH)^n$. Hence

$$N_1 < (2AH)^n \prod_{r=1}^{n}\left(\frac{nkp_2^r}{p_1^{r-1}}+1\right) < (4AH)^n \prod_{r=1}^{n}\frac{nkp_2^r}{p_1^{r-1}}$$
$$= (4nkAH)^n p_2^{\frac{1}{2}n(n+1)} p_1^{-\frac{1}{2}n(n-1)},$$

provided that $nkp_2^r \geqslant p_1^{r-1}$, etc. Hence

$$N < (4nkAH)^{nk}(p_2 \ldots p_k)^{\frac{1}{2}n(n+1)}(p_1 \ldots p_{k-1})^{-\frac{1}{2}n(n-1)}.n^n$$
$$= n^n(4nkAH)^{nk}(p_1 \ldots p_k)^n p_1^{-\frac{1}{2}n(n+1)} p_k^{\frac{1}{2}n(n-1)}.$$

Now for given $h_1,\ldots, h_b$ the number of values of $V_{m,r}^{(2)}$ does not exceed $(p_m/H)^{b-n}$; the number of sets of values of U_r' is $\prod_{m=1}^{k}(p_m/H)^b$; and the number of sets of h's is H^{bk}. Hence for given z_r the number of sets of values of W_r does not exceed

$$(p_1 \ldots p_k H^{-k})^{2b-n}H^{bk} = (p_1 \ldots p_k)^{2b-n}H^{-(b-n)k}.$$

Hence

$$K(z_1,\ldots,z_n) \leqslant n^n(4nkA)^{nk}H^{-(b-2n)k}p_1^{-\frac{1}{2}n(n+1)}p_k^{\frac{1}{2}n(n-1)}(p_1 \ldots p_k)^{2b}.$$

Hence

$$\sum_{y=1}^{P}\sum\nolimits_{2,1} |C_1 \ldots C_k|^2 \leqslant R_1|a|^{-\frac{1}{2}}P^{\frac{1}{2}}\log^{\frac{1}{2}}P.H^{-(b-2n)k}p_1^{-\frac{1}{2}n}p_k^{\frac{1}{2}n(n-1)}(p_1 \ldots p_k)^{2b},$$

where $$R_1 = 2(2bk)^n n^n(4Ank)^{nk}.$$

Also by (4.2)

$$\sum_{y=1}^{P}\sum\nolimits_{2,2} |C_1 \ldots C_k|^2 \leqslant R_2 PH^{-bk-(b-n-1)/n}(p_1 \ldots p_k)^{2b},$$

where $$R_2 = k.b!\{3.2^{4n-3}(n-1)\}^{b-n}.$$

Hence by (3.1)

$$\sum_{y=1}^{P}|S_1 \ldots S_k|^{2b} \leqslant R(p_1 \ldots p_k)^{2b}(|a|^{-\frac{1}{2}}P^{\frac{1}{2}}\log^{\frac{1}{2}}P.H^{2nk}p_1^{-\frac{1}{2}n}p_k^{\frac{1}{2}n(n-1)}+$$
$$+PH^{-(b-n-1)/n}),$$

where $$R = (b!)^{2k}\max(R_1, R_2).$$

Hence by (2.1)

$$|S| \leqslant (2R)^{\frac{1}{2bk}}P\Big\{(P^{-1}|a|^{-1}\log P.p_1^{-n}p_k^{n(n-1)})^{\frac{1}{4bk}}H^{\frac{n}{b}}+H^{-\frac{b-n-1}{2bkn}}\Big\}+O(p_1).$$

8. Choice of the parameters. We take

$$H = \left[P^{\frac{1}{8kn}}\right], \qquad p_k = \left[(1/a)^{\frac{1}{n}\left(\frac{n-1}{n}\right)^{k-1}}P^{-\frac{1}{8kn}}\right]H,$$

and

$$p_m/p_{m+1} = q_{m+1} = \left[(1/|a|)^{\frac{1}{n}\left\{\left(\frac{n-1}{n}\right)^{m-1}-\left(\frac{n-1}{n}\right)^{m}\right\}}\right] \quad (m = 1,\ldots, k-1).$$

To make $p_k \geqslant H$ we require

$$(k-1)\log\frac{n}{n-1} \leqslant \log 8k+\log\frac{\log 1/|a|}{\log P}.$$

This is satisfied by taking

$$k = \left[\left(\log 8n+\log\frac{\log 1/|a|}{\log P}\right)\Big/\log\frac{n}{n-1}\right]+1$$

since this makes $k > n$.

We obtain $$p_1 = p_k q_2 \cdots q_k \leqslant (1/|a|)^{1/n}$$

as required in § 5. Also clearly

$$p_1 \geqslant 2^{-k}(1/|a|)^{1/n},$$

so that $$|a|^{-(1/4bk)} p_1^{-(n/4bk)} \leqslant 2^{n/4b} < 2.$$

Also

$$p_m^{\frac{1}{n-1}} \geqslant \left\{\frac{1}{2^k}\left(\frac{1}{|a|}\right)^{\frac{1}{n}\left(\frac{n-1}{n}\right)^{m-1}}\right\}^{\frac{1}{n-1}} = \left(\frac{1}{2}\right)^{\frac{k}{n-1}}\left(\frac{1}{|a|}\right)^{\frac{1}{n^2}\left(\frac{n-1}{n}\right)^{m-2}} \geqslant \left(\frac{1}{2}\right)^{\frac{k}{n-2}} q_m,$$

and the conditions $nkp_{m+1}^r \geqslant p_m^{r-1}$ are satisfied. It is also easily verified that

$$p_k \leqslant p^{1/8n^2}.$$

Hence we obtain

$$S = O\left(b.P^{1-\frac{1}{4bk}}\log^{\frac{1}{4bk}}P.P^{\frac{1}{8bk}}.P^{\frac{n-1}{32bkn}}\right)+O\left(bP^{1-\frac{b-n-1}{16bkn^2}}\right)+O\left\{(1/|a|)^{\frac{1}{n}}\right\}.$$

We may take $b = 2n$; the first term is small compared with the second, and we obtain

$$S = O\left(nP^{1-\frac{n-1}{32k^2n^3}}\right)+O\left\{(1/|a|)^{\frac{1}{n}}\right\}.$$

If $P \geqslant (1/|a|)^{1/n}$, then $\log(1/|a|) \leqslant n\log P$, and $k < An\log n$. If $P < (1/|a|)^{1/n}$, the second term makes the result trivial. Hence in any case

$$S = O\left(nP^{1-\frac{A}{n^4\log^2 n}}\right)+O\left\{(1/|a|)^{\frac{1}{n}}\right\}, \tag{8.1}$$

where A is an absolute constant. The only restriction is

$$2(n+1)|a|P \leqslant 1.$$

9. Application to $\zeta(s)$. Let $0 < \sigma \leqslant 1$, $N \leqslant N' \leqslant 2N$. Then

$$\sum_N^{N'} m^{-s} = O\left(N^{-\sigma}\max_{N \leqslant N'' \leqslant 2N}\left|\sum_N^{N''} e^{-it\log m}\right|\right).$$

Divide the sum on the right into $O(N/\mu)$ parts, each of length not exceeding μ, where

$$\mu = \tfrac{1}{2}Nt^{-1/(n+2)}.$$

If one of these new sums is $\sum\limits_{Q+1}^{Q+P}$, then

$$\left|\sum_{Q+1}^{Q+P} e^{-it\log m}\right| = \left|\sum_{r=1}^{P} e^{-it\log(Q+r)}\right|$$

$$= \left|\sum_{r=1}^{P} \exp\left\{-it\left(\frac{r}{Q}+\dots+\frac{(-1)^n r^{n+1}}{(n+1)Q^{n+1}}\right)-it\left(\frac{(-1)^{n+1}r^{n+2}}{(n+2)Q^{n+2}}+\dots\right)\right\}\right|$$

$$= \left|\sum_{r=1}^{P} \exp\left\{-it\left(\frac{r}{Q}+\dots+\frac{(-1)^n r^{n+1}}{(n+1)Q^{n+1}}\right)\right\}\sum_{\nu=0}^{\infty} e_\nu(t)\left(\frac{r}{Q}\right)^\nu\right|, \quad \text{say,}$$

$$= \left|\sum_{\nu=0}^{\infty} \frac{e_\nu(t)}{Q^\nu}\sum_{r=1}^{P} r^\nu \exp\left\{-it\left(\frac{r}{Q}+\dots+\frac{(-1)^n r^{n+1}}{(n+1)Q^{n+1}}\right)\right\}\right|$$

$$= O\left[\sum_{\nu=0}^{\infty} |e_\nu(t)|\left(\frac{P}{Q}\right)^\nu \max_{1\leqslant P'\leqslant P}\left|\sum_{r=1}^{P'} \exp\left\{-it\left(\frac{r}{Q}+\dots+\frac{(-1)^n r^{n+1}}{(n+1)Q^{n+1}}\right)\right\}\right|\right].$$

We apply the above result to the inner sum, say S. Here

$$|a| = \frac{t}{2\pi(n+1)Q^{n+1}}.$$

Since $P' \leqslant P \leqslant \mu$, we obtain

$$S = O(n\mu^{1-An^{-4}\log^{-2}n})+O\{(Q^{n+1}t^{-1})^{1/n}\}$$

$$= O\{n(Nt^{-1/(n+2)})^{1-An^{-4}\log^{-2}n}\}+O(N^{1+(1/n)}t^{-(1/n)}).$$

Also $$\sum_{\nu=0}^{\infty} |e_\nu(t)|\left(\frac{P}{Q}\right)^\nu \leqslant \exp\left\{t\left(\frac{\mu^{n+2}}{(n+2)Q^{n+2}}+\dots\right)\right\} = O(1).$$

Hence

$$\sum_{N}^{N'} n^{-s} = O\left(\frac{N^{1-\sigma}}{\mu}\max|S|\right) = O\left[nN^{1-\sigma}\left\{\left(\frac{N}{t^{1/(n+2)}}\right)^{-An^{-4}\log^{-2}n}+\frac{N^{1/n}}{t^{2/n(n+2)}}\right\}\right].$$

Let $n = [\frac{3}{2}\log t/\log N]$. It is then easily verified that

$$\log\left(\frac{N}{t^{1/(n+2)}}\right)^{-An^{-4}\log^{-2}n} < -A\frac{\log^5 N}{\log^4 t}\frac{1}{\log^2(\log t/\log N)},$$

and $$\log\{N^{1/n}/t^{2/n(n+2)}\} < -A\log^2 N/\log t.$$

The second term can be omitted if $N<t$. Also the condition $2(n+1)|a|P \leqslant 1$ is now $tP \leqslant \pi Q^{n+1}$, which is true if $t \leqslant N^n$, i.e. $n \geqslant \log t/\log N$. We have thus proved that

$$\sum_{N}^{N'} m^{-s} = O\left\{\frac{\log t}{\log N}\cdot N^{1-\sigma}\exp\left(-A\frac{\log^5 N}{\log^4 t}\cdot\frac{1}{\log^2(\log t/\log N)}\right)\right\}$$

$$= O\left[\frac{\log t}{\log N}\cdot N^{1-\sigma}\exp\left\{-A\log N\left(\frac{\log N}{\log t}\right)^{4+\epsilon}\right\}\right] \tag{9.1}$$

for every positive ϵ. This result is non-trivial only if

$$N > \exp(\log t)^{\frac{4+\epsilon}{5+\epsilon}} = N_0.$$

We can also write it

$$\sum_{N}^{N'} m^{-s} = O\left[\frac{\log t}{\log N}.\exp\left\{(1-\sigma)\log N - A\log N\left(\frac{\log N}{\log t}\right)^{4+\epsilon}\right\}\right].$$

If $\sigma < 1$, the exponential, considered as a function of $\log N$, is a maximum if

$$\log N = \left(\frac{1-\sigma}{(5+\epsilon)A}\right)^{\frac{1}{4+\epsilon}} \log t.$$

Hence, with new A,

$$\sum_{N}^{N'} m^{-s} = O(\log^{\frac{1}{5}}t.t^{A(1-\sigma)^{\alpha}})$$

uniformly in $\sigma < 1$, where $\alpha = (5+\epsilon)/(4+\epsilon)$. Hence for $\sigma < 1$

$$\begin{aligned}\zeta(s) &= \sum_{m \leqslant N_0} m^{-s} + O(\log^{\frac{6}{5}}t.t^{A(1-\sigma)^{\alpha}}) \\ &= O\left(\frac{1}{1-\sigma}e^{(1-\sigma)(\log t)^{\alpha}}\right) + O(\log^{\frac{6}{5}}t.t^{A(1-\sigma)^{\alpha}}). \qquad (9.2)\end{aligned}$$

For a fixed $\sigma < 1$ the first term may, of course, be omitted. The result obtained is an improvement on what was previously known for sufficiently small values of $1-\sigma$.

Taking $$\sigma = 1-(\log t)^{-\frac{4+\epsilon}{5+\epsilon}}$$
we obtain $$\zeta(s) = O(\log^{\frac{6}{5}}t). \qquad (9.3)$$

By the usual Phragmén-Lindelöf argument, this result therefore holds uniformly for $\sigma \geqslant 1-(\log t)^{-\frac{4+\epsilon}{5+\epsilon}}$.

For $\sigma = 1$, (9.1) gives

$$\sum_{N}^{N'} m^{-1-it} = O\left[\frac{\log t}{\log N}.\exp\left\{-A\log N\left(\frac{\log N}{\log t}\right)^{4+\epsilon}\right\}\right]$$

for $N \geqslant N_0$. Let $N \geqslant \exp(\log t)^{\frac{4+2\epsilon}{5+2\epsilon}} = N_1$. Then

$$\log^{5+\epsilon}N/\log^{4+\epsilon}t \geqslant (\log t)^{\frac{\epsilon}{5+2\epsilon}}.$$

Hence $$\sum_{N_1 \leqslant m < t} m^{-1-it} = O\left\{\log^{\frac{6}{5}}t.\exp\left[-A(\log t)^{\frac{\epsilon}{5+2\epsilon}}\right]\right\} = O(1).$$

Also $$\sum_{m<N_1} m^{-1-it} = O(\log N_1) = O\left\{(\log t)^{\frac{4+2\epsilon}{5+2\epsilon}}\right\}.$$

Hence $$\zeta(1+it) = O\{(\log t)^{\frac{4}{5}+\delta}\} \tag{9.4}$$
for every positive δ.

10. The zeros of $\zeta(s)$. We shall use the following theorem.

THEOREM. *Let $\zeta(s) = O(e^{\phi(t)})$ for $\sigma \geqslant 1-\theta(t)$, where $\phi(t)$ and $1/\theta(t)$ are positive non-decreasing functions of t such that $\phi(2t) = O\{\phi(t)\}$, $\theta(t) = O\{\theta(2t)\}$, $1/\theta(t) = O(e^{A\phi(t)})$. Then there is a domain*
$$\sigma > 1-A\,\theta(t)/\phi(t) \quad (t > t_0)$$
in which $\zeta(s)$ has no zeros.

This is the obvious generalization of Theorems 8 and 13 of Titchmarsh (**8**).

Suppose that $\beta+i\gamma$ ($\frac{1}{2} < \beta < 1$, $\gamma > 0$) is a zero of $\zeta(s)$. Let
$$1+e^{-A\phi(\gamma)} \leqslant \sigma_0 \leqslant 2, \qquad r = A_1\,\theta(\gamma),$$
$$s_0 = \sigma_0+i\gamma, \qquad s_0' = \sigma_0+2i\gamma.$$
Then, if A_1 is properly chosen, the circles $|s-s_0| \leqslant r$, $|s-s_0'| \leqslant r$, both lie entirely in the region $\sigma \geqslant 1-\theta(t)$. Hence in the first circle
$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right| < \frac{A\,|\zeta(s)|}{\sigma_0-1} < Ae^{A\phi(\gamma)}.$$
If $\beta > \sigma_0-\frac{1}{2}r$, it follows from the second lemma of § 1.41 of Titchmarsh (**8**) that
$$-\mathbf{R}\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)} < A\frac{\phi(\gamma)}{\theta(\gamma)} - \frac{1}{\sigma_0-\beta}.$$
Similarly, $$-\mathbf{R}\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)} < A\frac{\phi(\gamma)}{\theta(\gamma)}.$$
Also $$-\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} < \frac{5}{4}\,\frac{1}{\sigma_0-1}$$
if σ_0-1 is small enough. Combining these inequalities with the known inequality
$$-3\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} - 4\mathbf{R}\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)} - \mathbf{R}\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)} \geqslant 0$$
and solving for $1-\beta$, we obtain
$$1-\beta \geqslant \left\{\frac{1}{4} - \frac{A_2(\sigma_0-1)\phi(\gamma)}{\theta(\gamma)}\right\} \Big/ \left\{\frac{15}{4}\,\frac{1}{\sigma_0-1} + A_3\frac{\phi(\gamma)}{\theta(\gamma)}\right\}.$$
To make the right-hand side positive we take $\sigma_0 = 1+\frac{1}{5}\theta(\gamma)/A_2\,\phi(\gamma)$; we obtain
$$1-\beta \geqslant A\,\theta(\gamma)/\phi(\gamma)$$

as stated. The alternative is

$$\beta \leqslant \sigma_0 - \tfrac{1}{2}r = 1 + A\,\theta(\gamma)/\phi(\gamma) - A\,\theta(\gamma),$$

which gives
$$1-\beta \geqslant A\,\theta(\gamma) > A\,\theta(\gamma)/\phi(\gamma)$$
again.

By the result of the previous section we can take

$$\theta(t) = (\log t)^{-\frac{4+\epsilon}{5+\epsilon}}, \qquad \phi(t) = \tfrac{6}{5}\log\log t.$$

It follows that *for every positive* δ, *there is a* $K = K(\delta)$ *such that* $\zeta(s)$ *has no zeros in the region*

$$\sigma > 1 - K(\log t)^{-(\frac{4}{5}+\delta)}, \qquad t > t_0.$$

Inequalities for $1/\zeta(s)$ and $\zeta'(s)/\zeta(s)$ are easily deduced by standard methods; e.g. $1/\zeta(1+it)$ and $\zeta'(1+it)/\zeta(1+it)$ are both of the form $O\{(\log t)^{\frac{4}{5}+\delta}\}$.

11. Application to $\pi(x)$. It is proved by Ingham (**7**) (Theorem 22) that, if $\zeta(s)$ has no zeros in the domain $\sigma > 1-\eta(|t|)$, where $\eta(t)$ satisfies certain simple conditions, and $\omega(x)$ is the minimum of $\eta(t)\log x + \log t$ for $t \geqslant 1$, then

$$\pi(x) = \operatorname{li} x + O(xe^{-A\omega(x)})$$

for some constant A. We can now take

$$\eta(t) = K(\log t)^{-\frac{4}{5}-\delta}, \qquad \omega(x) = K(\log x)^{\frac{5}{9+5\delta}}.$$

Hence
$$\pi(x) = \operatorname{li} x + O(xe^{-A(\log x)^{\frac{5}{9}-\epsilon}})$$
for every positive ϵ.

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THE APPROXIMATE FUNCTIONAL EQUATION FOR $\zeta^2(s)$

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1. It was proved by Hardy and Littlewood* that, if

$$-\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{3}{2}, \qquad x > A, \qquad y > A, \qquad xy = \left(\frac{t}{2\pi}\right)^2,$$

then

$$\zeta^2(s) = \sum_{n \leqslant x} \frac{d(n)}{n^s} + \chi^2(s) \sum_{n \leqslant y} \frac{d(n)}{n^{1-s}} + O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{|t|}\right)^{\frac{1}{4}} \log|t|\right\},$$

where
$$\chi(s) = 2^s \pi^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s).$$

In the present paper I give an alternative proof of this theorem; the O-term is replaced by $O(x^{\frac{1}{2}-\sigma}\log t)$, which is equivalent to the previous one in the most interesting case $x = y = t/(2\pi)$. The main difference between the proofs is that, while Hardy and Littlewood use an integrated formula, and end with a differencing argument, my proof is a direct extension of the Hardy-Littlewood method for $\zeta(s)$.

2. We first obtain an exact formula for $\zeta^2(s)$. If $\sigma > -\frac{1}{4}$,

$$\zeta^2(s) = \sum_{n \leqslant x} \frac{d(n)}{n^s} - x^{-s} \sum_{n \leqslant x} d(n) + \frac{2s-s^2}{(s-1)^2} x^{1-s} + \frac{s}{s-1} x^{1-s}(2\gamma + \log x) +$$
$$+ \tfrac{1}{4}x^{-s} - 2^{4s}\pi^{2s-2} s \sum_{n=1}^{\infty} \frac{d(n)}{n^{1-s}} \int_{4\pi\sqrt{(nx)}}^{\infty} \frac{K_1(v) + \frac{1}{2}\pi Y_1(v)}{v^{2s}}\, dv, \tag{2.1}$$

where $K_1(v)$ and $Y_1(v)$ are the Bessel functions usually so denoted. To prove this, suppose first that $\sigma > 1$, and consider the integral

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s x^{w-s}}{w(s-w)} \zeta^2(w)\, dw \quad (1 < c < \sigma).$$

Inserting the series $\sum d(n) n^{-w}$ for $\zeta^2(w)$, and integrating term by

* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2) 29 (1929), 81–97.

term, we obtain

$$I = \frac{1}{2\pi i}\sum_{n=1}^{\infty} d(n)\int_{c-i\infty}^{c+i\infty} \frac{sx^{w-s}}{w(s-w)} n^{-w}\,dw = \sum_{n\leqslant x} d(n)x^{-s} + \sum_{n>x} d(n)n^{-s}$$

$$= \zeta^2(s) - \sum_{n\leqslant x} d(n)n^{-s} + x^{-s}\sum_{n\leqslant x} d(n).$$

On the other hand, if we move the contour in I to $\mathbf{R}(w) = -b < 0$, we obtain

$$I = \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} \frac{sx^{w-s}}{w(s-w)}\zeta^2(w)\,dw + \frac{2s-s^2}{(s-1)^2}x^{1-s} +$$

$$+\frac{s}{s-1}x^{1-s}(2\gamma+\log x)+\tfrac{1}{4}x^{-s}.$$

The first term is

$$\frac{1}{2\pi i}\int_{1+b-i\infty}^{1+b+i\infty} \frac{sx^{1-w-s}}{(1-w)(s-1+w)}\zeta^2(1-w)\,dw$$

$$= \frac{2}{\pi i}\int_{1+b-i\infty}^{1+b+i\infty} \frac{sx^{1-w-s}}{(1-w)(s-1+w)}(2\pi)^{-2w}\Gamma^2(w)\cos^2\tfrac{1}{2}w\pi\,\zeta^2(w)\,dw$$

$$= \frac{2}{\pi i}\sum_{n=1}^{\infty} d(n)\int_{1+b-i\infty}^{1+b+i\infty} \frac{sx^{1-w-s}}{(1-w)(s-1+w)}\Gamma^2(w)\cos^2\tfrac{1}{2}w\pi(2\pi\sqrt{n})^{-2w}\,dw.$$

Now we have the Mellin transforms*

$$\xi^{-1}\{K_1(\xi)+\tfrac{1}{2}\pi Y_1(\xi)\}, \qquad 2^{z-2}\Gamma(\tfrac{1}{2}z)\Gamma(\tfrac{1}{2}z-1)\cos^2\tfrac{1}{4}z\pi.$$

Thus, if $a > 0$,

$$\frac{K_1(\xi)+\frac{1}{2}\pi Y_1(\xi)}{\xi} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} 2^{z-2}\Gamma(\tfrac{1}{2}z)\Gamma(\tfrac{1}{2}z-1)\cos^2\tfrac{1}{4}z\pi\,\xi^{-z}\,dz$$

$$= \frac{1}{\pi i}\int_{\frac{1}{2}a-i\infty}^{\frac{1}{2}a+i\infty} 2^{2w-2}\Gamma(w)\Gamma(w-1)\cos^2\tfrac{1}{2}w\pi\,\xi^{-2w}\,dw.$$

Multiplying by ξ^{1-2s}, and integrating over $4\pi\sqrt{(nx)}$, ∞ (or using

* See my *Fourier Integrals*, (7.9.8) and (7.9.11).

Parseval's formula for Mellin transforms), we obtain

$$\int\limits_{4\pi\sqrt{(nx)}}^{\infty} \frac{K_1(\xi)+\frac{1}{2}\pi Y_1(\xi)}{\xi^{2s}}\,d\xi$$

$$= -\frac{1}{\pi i}\int\limits_{\frac{1}{2}a-i\infty}^{\frac{1}{2}a+i\infty} \frac{2^{2w-2}\Gamma(w)\Gamma(w-1)\cos^2\frac{1}{2}w\pi\{4\pi\sqrt{(nx)}\}^{2-2s-2w}}{2-2s-2w}\,dw.$$

The result now follows for $\sigma > 1$, and so by analytic continuation for $\sigma > -\frac{1}{4}$.

3. We now take $0 \leqslant \sigma \leqslant 1$ (there would be little difficulty in widening the range) and $t > 0$, and prove

$$\zeta^2(s) = \sum_{n\leqslant x}\frac{d(n)}{n^s}+\chi^2(s)\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}}+O(x^{\frac{1}{2}-\sigma}\log t). \tag{3.1}$$

We may suppose without loss of generality that $x \geqslant y$; for, on dividing by $\chi^2(s)$ ($|\chi(s)| \sim Kt^{\frac{1}{2}-\sigma}$) and then changing s into $1-s$, we obtain the corresponding result with x and y interchanged.

We know that

$$\sum_{n\leqslant x} d(n) = x\log x+(2\gamma-1)x+O(x^{\frac{1}{2}}). \tag{3.2}$$

Hence the second, third, and fourth terms on the right of (2.1) are together

$$O(x^{\frac{1}{2}-\sigma})+O(x^{1-\sigma}t^{-1}\log t) = O(x^{\frac{1}{2}-\sigma}\log t),$$

since $y > A$, $x < At^2$. Now

$$K_1(v)+\tfrac{1}{2}\pi Y_1(v) = \left(\frac{\pi}{2v}\right)^{\frac{1}{2}}\left\{-\cos(v-\tfrac{1}{4}\pi)+\frac{3}{8v}\sin(v-\tfrac{1}{4}\pi)+O\left(\frac{1}{v^2}\right)\right\},$$

and the contribution of the O-term is

$$O\left\{t\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-\sigma}}\int\limits_{4\pi\sqrt{(nx)}}^{\infty}\frac{dv}{v^{2\sigma+\frac{5}{2}}}\right\} = O\left\{t\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-\sigma}(nx)^{\sigma+\frac{3}{4}}}\right\}$$

$$= O(tx^{-\sigma-\frac{3}{4}}) = O(x^{\frac{1}{4}-\sigma}).$$

The leading term contributes

$$2^{4s-\frac{1}{2}}\pi^{2s-\frac{3}{2}}s\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-s}}\int\limits_{4\pi\sqrt{(nx)}}^{\infty}\frac{\cos(v-\frac{1}{4}\pi)}{v^{2s+\frac{1}{2}}}\,dv. \tag{3.3}$$

In this sum we write

$$\sum_{n>y} = \sum_{y<n<y+\sqrt{y}} + \sum_{y+\sqrt{y}\leqslant n<4y} + \sum_{n>4y} = \Sigma_1+\Sigma_2+\Sigma_3.$$

We now use (2.25), (2.26), (2.27) of Hardy and Littlewood's Lemma β

(it being easily seen that these are true without the restriction $\sigma \leqslant \frac{3}{2}$ stated), and the corresponding results with sine for cosine. From (2.27) it follows that in $\sum_1$

$$\int_{4\pi\sqrt{(nx)}}^{\infty} \frac{\cos(v-\frac{1}{4}\pi)}{v^{2s+\frac{1}{2}}}\,dv = O\{(\sqrt{(nx)})^{-2\sigma-\frac{1}{2}}t^{\frac{1}{2}}\} = O(t^{-2\sigma}),$$

and hence

$$\sum\nolimits_1 = O\Big(\sum_{y<n<y+\sqrt{y}} d(n)n^{\sigma-1}t^{-2\sigma}\Big) = O\Big(y^{\sigma-1}t^{-2\sigma} \sum_{y<n<y+\sqrt{y}} d(n)\Big)$$
$$= O(y^{\sigma-1}t^{-2\sigma}y^{\frac{1}{2}}\log y) = O(x^{\frac{1}{2}-\sigma}t^{-1}\log t)$$

as required.

Next, by (2.26), the integral in $\sum_3$ is $O\{(nx)^{-\sigma-\frac{1}{4}}\}$, and hence

$$\sum\nolimits_3 = O\Big\{\sum_{n>4y} \frac{d(n)}{n^{1-\sigma}}(nx)^{-\sigma-\frac{1}{4}}\Big\} = O\Big\{x^{-\sigma-\frac{1}{4}} \sum_{n>4y} \frac{d(n)}{n^{\frac{5}{4}}}\Big\}$$
$$= O(x^{-\sigma-\frac{1}{4}}y^{-\frac{1}{4}}\log y) = O(x^{-\sigma}t^{-\frac{1}{2}}\log t) = O(x^{\frac{1}{2}-\sigma}t^{-1}\log t).$$

In $\sum_2$ we use (2.25), which gives

$$\int_{4\pi\sqrt{(nx)}}^{\infty} \frac{\cos(v-\frac{1}{4}\pi)}{v^{2s+\frac{1}{2}}}\,dv = \frac{ie^{i\{4\pi\sqrt{(nx)}-\frac{1}{4}\pi\}}}{2\{4\pi\sqrt{(nx)}-2t\}\{4\pi\sqrt{(nx)}\}^{2s-\frac{1}{2}}}+$$
$$+O\{(nx)^{-\sigma-\frac{1}{4}}\}+O\Big\{\frac{t(nx)^{\frac{1}{4}-\sigma}}{(4\pi\sqrt{(nx)}-2t)^3}\Big\}.$$

The first O-term gives $O(x^{\frac{1}{2}-\sigma}t^{-1}\log t)$ as in $\sum_3$. The second contributes to $\sum_2$

$$O\Big\{y^{\sigma-1}t^{\frac{3}{2}-2\sigma} \sum_{y+\sqrt{y}\leqslant n<4y} \frac{d(n)}{(2\pi\sqrt{(nx)}-t)^3}\Big\}.$$

Now

$$(2\pi\sqrt{(nx)}-t)^{-3} = (2\pi\sqrt{x})^{-3}(\sqrt{n}-\sqrt{y})^{-3} = O\{(y/x)^{\frac{3}{2}}(n-y)^{-3}\},$$

and

$$\sum_{y+\sqrt{y}\leqslant n<4y} \frac{d(n)}{(n-y)^3} = O\Big\{\sum_k \sum_{y+k\sqrt{y}\leqslant n<y+(k+1)\sqrt{y}} \frac{d(n)}{(k\sqrt{y})^3}\Big\}$$
$$= O\Big\{\sum_k \frac{\sqrt{y}\log y}{(k\sqrt{y})^3}\Big\} = O(y^{-1}\log y).$$

Altogether we get

$$O(y^{\sigma-\frac{1}{2}}t^{\frac{3}{2}-2\sigma}x^{-\frac{3}{2}}\log t) = O(x^{-\sigma-1}t^{\frac{1}{2}}\log t) = O(x^{\frac{1}{2}-\sigma}t^{-1}\log t).$$

4. The remaining sum is a bounded multiple of

$$x^{-s-\frac{1}{4}} \sum_{y+\sqrt{y}\leqslant n<4y} \frac{d(n)e^{4\pi i\sqrt{(nx)}}}{n^{\frac{3}{4}}(\sqrt{n}-\sqrt{y})}.$$

It is sufficient to prove that

$$S = \sum_{y+\sqrt{y}\leqslant n<N} \frac{d(n)e^{4\pi i\sqrt{(nx)}}}{n-y} = O(x^{\frac{1}{4}}y^{-\frac{1}{4}}\log t) \quad (N < 4y),$$

for, by partial summation, the above sum is then

$$O(x^{-\sigma}y^{-\frac{1}{2}}\log t) = O(x^{\frac{1}{2}-\sigma}t^{-1}\log t).$$

The difficulty is that, for $x = y$, the sum obtained by replacing every term of S by its modulus is only $O(\log^2 t)$.

Let
$$S' = \sum_{k=1}^{K} \sum_{y+k\sqrt{y}\leqslant n<y+(k+1)\sqrt{y}} \frac{d(n)e^{4\pi i\sqrt{(nx)}}}{k\sqrt{y}},$$
where $K = [(N-y)y^{-\frac{1}{2}}]-1$. Then

$$|S-S'| \leqslant \sum_{k=1}^{K} \sum_{y+k\sqrt{y}\leqslant n<y+(k+1)\sqrt{y}} \frac{d(n)}{k^2\sqrt{y}} + \sum_{y+(K+1)\sqrt{y}\leqslant n<y+(K+2)\sqrt{y}} \frac{d(n)}{(K+1)\sqrt{y}}$$
$$= \sum_{k=1}^{K} \frac{O(\sqrt{y}\log y)}{k^2\sqrt{y}} + \frac{O(\sqrt{y}\log y)}{K\sqrt{y}} = O(\log y).$$

Let
$$c_k = \sum_{y+k\sqrt{y}\leqslant n<y+(k+1)\sqrt{y}} d(n)e^{4\pi i\sqrt{(nx)}}, \qquad C_k = c_1+c_2+\ldots+c_k.$$

Then
$$S' = \frac{1}{\sqrt{y}} \sum_{k=1}^{K} \frac{c_k}{k} = \frac{1}{\sqrt{y}} \left\{ \sum_{k=1}^{K-1} \frac{C_k}{k(k+1)} + \frac{C_K}{K} \right\}.$$

Now
$$C_k = \sum_{y+\sqrt{y}\leqslant n<y+(k+1)\sqrt{y}} d(n)e^{4\pi i\sqrt{(nx)}} = \sum_{y+\sqrt{y}\leqslant \mu\nu<y+(k+1)\sqrt{y}} \sum e^{4\pi i\sqrt{(\mu\nu x)}}$$
$$= 2\sum_{\mu<\nu}\sum e^{4\pi i\sqrt{(\mu\nu x)}} + O(\sqrt{y}).$$

The ν-sum is of the form $\sum e^{2\pi i f(\nu)}$, where

$$f(\nu) = 2\sqrt{(\mu\nu x)}, \qquad f''(\nu) = -\tfrac{1}{2}(\mu x)^{\frac{1}{2}}\nu^{-\frac{3}{2}}.$$

Since $y/\mu \leqslant \nu \leqslant 4y/\mu$, we have

$$\tfrac{1}{16}\mu^2 x^{\frac{1}{2}}y^{-\frac{3}{2}} \leqslant |f''(\nu)| \leqslant \tfrac{1}{2}\mu^2 x^{\frac{1}{2}}y^{-\frac{3}{2}}.$$

Hence*
$$\sum e^{2\pi i f(\nu)} = O\left(\frac{ky^{\frac{1}{2}}}{\mu} \frac{\mu x^{\frac{1}{4}}}{y^{\frac{3}{4}}}\right) + O\left(\frac{y^{\frac{3}{4}}}{\mu x^{\frac{1}{4}}}\right)$$
$$= O(kx^{\frac{1}{4}}y^{-\frac{1}{4}}) + O(\mu^{-1}x^{-\frac{1}{4}}y^{\frac{3}{4}}).$$

Since μ varies at most from 1 to $2\sqrt{y}$, it follows that

$$C_k = O(kx^{\frac{1}{4}}y^{\frac{1}{4}}) + O(x^{-\frac{1}{4}}y^{\frac{3}{4}}\log y) + O(\sqrt{y}),$$

* See E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 2 (1931), 161–73, Theorem 1.

and the last term can be omitted in comparison with the first. Hence

$$S' = O\left(x^{\frac{1}{4}}y^{-\frac{1}{4}}\sum_{k=1}^{K-1}\frac{1}{k+1}\right)+O\left(x^{-\frac{1}{4}}y^{\frac{3}{4}}\log y\sum_{k=1}^{K-1}\frac{1}{k(k+1)}\right)+$$

$$+O(x^{\frac{1}{4}}y^{-\frac{1}{4}})+O(x^{-\frac{1}{4}}y^{\frac{3}{4}}\log y) = O(x^{\frac{1}{4}}y^{-\frac{1}{4}}\log t),$$

the required result.

5. We have still to discuss the part of (3.3) with $n \leqslant y$, but this is mainly a repetition of the above argument. We have

$$\int\limits_{4\pi\sqrt{(nx)}}^{\infty}\frac{\cos(v-\frac{1}{4}\pi)}{v^{2s+\frac{1}{2}}}\,dv$$

$$= \frac{\cos\{4\pi\sqrt{(nx)}-\frac{1}{4}\pi\}}{(2s-\frac{1}{2})\{4\pi\sqrt{(nx)}\}^{2s-\frac{1}{2}}}-\frac{\sin\{4\pi\sqrt{(nx)}-\frac{1}{4}\pi\}}{(2s-\frac{1}{2})(2s-\frac{3}{2})\{(4\pi\sqrt{(nx)}\}^{2s-\frac{3}{2}}}+$$

$$+\Gamma(\tfrac{1}{2}-2s)\cos\pi s+\frac{1}{(2s-\frac{1}{2})(2s-\frac{3}{2})}\int\limits_{0}^{4\pi\sqrt{(nx)}}\frac{\cos(v-\frac{1}{4}\pi)}{v^{2s-\frac{3}{2}}}\,dv$$

for $\sigma < \frac{5}{4}$. The first integrated term contributes

$$O\left\{\sum_{n<y}\frac{d(n)}{n^{1-\sigma}}\frac{1}{(nx)^{\sigma-\frac{1}{4}}}\right\} = O(x^{\frac{1}{4}-\sigma}y^{\frac{3}{4}}\log y) = O(x^{-\sigma}t^{\frac{1}{2}}\log t),$$

and similarly for the second integrated term. The Γ-function term contributes

$$2^{4s-\frac{1}{2}}\pi^{2s-\frac{3}{2}}s\Gamma(\tfrac{1}{2}-2s)\cos\pi s\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}} = \{\chi^2(s)+O(t^{-2\sigma})\}\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}},$$

and the O-term is $O(t^{-2\sigma}y^{\sigma}\log y) = O(x^{-\sigma}\log t)$. We write the remaining sum as

$$\sum_{n<\frac{1}{2}y}+\sum_{\frac{1}{2}y\leqslant n<y-\sqrt{y}}+\sum_{y-\sqrt{y}\leqslant n\leqslant y},$$

and apply the appropriate parts of the Hardy-Littlewood Lemma β, which extend easily enough to the required σ-range.

Finally, the term involving $v^{-1}\sin(v-\frac{1}{4}\pi)$ in the asymptotic expansion can be dealt with in the same way; but the extra v^{-1} makes most of the analysis comparatively trivial.

A CONVERGENCE CRITERION FOR FOURIER SERIES

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1. THE object of this note is to generalize a criterion, due to W. H. Young (**5**), and which is stated by Hardy and Littlewood (**1**) to be 'one of the most interesting of the more modern criteria' for the convergence of a Fourier series.

It is remarked by Hardy and Littlewood that W. H. Young's criterion may be given the following form: *In order that the Fourier series of an integrable function $f(t)$ converge to s at $t = x$, it is sufficient that the function*

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$$

fulfil for small t the conditions

$$\phi(t) = o(1) \quad \textit{and} \quad V(\phi;[t,2t]) = O(1), \tag{1.1}$$

where $V(\phi;[\alpha,\beta])$ denotes the total variation of ϕ in any interval $[\alpha,\beta]$.

The generalized criterion that I propose to establish here consists simply in replacing in the above statement *the total variation* by its pth power and by its exponential generalizations.† In the pth power case I shall prove a slightly more complete statement, Theorem 1 below, containing also certain refinements on Hardy and Littlewood (**1**).

2. Direct proof of the pth-power generalization

In the sequel we shall write $f(t)$ in place of $\phi(t)$. This amounts to supposing f even, $x = 0$, and $s = 0$, and is clearly no loss of generality. *We shall show that* (1.1) *may be replaced by*

$$f(t) = o(1) \quad \textit{and} \quad V_p(f;[t,2t]) = O(1), \tag{2.1}$$

where $V_p^p(f;[\alpha,\beta])$ denotes, as in Young (**3**), the upper bound of the sum

$$\sum |\Delta f|^p$$

for all systems of non-overlapping sub-intervals Δ of $[\alpha,\beta]$.

We observe at once that we may suppose, by increasing the exponent p a little, that f fulfils the more stringent condition

$$|f(t)|+V_p(f;[t,2t]) = o(1) \tag{2.2}$$

† Wiener (**2**), L. C. Young (**3**), (**4**).

for small t. This is an immediate consequence of the elementary inequality (8.2a) of (**3**), 259. We may suppose further that $p > 1$.

We have now only to prove that (2.2) implies the convergence to zero of the Fourier series of f at $x = 0$. By the 'localization principle' it is enough to show that the inequality

$$|f(t)|+V_p(f;[t,2t]) \leqslant \epsilon \quad \textit{for all } t \textit{ of } [0,\pi], \tag{2.3}$$

implies $|s_n| \leqslant K\epsilon$, where s_n denotes the sum to $2n+1$ terms of the Fourier series of f at $x = 0$, and K is a constant independent of ϵ and n.

For this purpose, let us choose $q > 1$ so that

$$\frac{1}{p}+\frac{1}{q} > 1. \tag{2.4}$$

We write as usual

$$s_n = \frac{1}{\pi}\int\limits_0^\pi f(t)\, dg_n(t) \tag{2.5}$$

where

$$g_n(t) = \int\limits_0^t \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt. \tag{2.6}$$

We observe, as in (**3**) § 12, 274, that, since $q > 1$, there exists a constant K such that

$$V_q^q(g_n;[\alpha,\beta]) \leqslant K^q \sum_{n\alpha/\pi-1<m<n\beta/\pi+1} (m+1)^{-q}$$

for any sub-interval $[\alpha,\beta]$ of $[0,\pi]$. In particular, if $2^{\nu+1} < n$,

$$V_q(g_n;[2^\nu\pi/n, 2^{\nu+1}\pi/n]) \leqslant K\Big\{\sum_{2^\nu\leqslant m}(m+1)^{-q}\Big\}^{1/q} \leqslant K2^{-\nu(1-1/q)}. \tag{2.7}$$

Thus, by the main inequality (10.9) of (**3**), 266, the absolute value of the integral

$$I_\nu = \int\limits_{2^\nu\pi/n}^{2^{\nu+1}\pi/n} f\, dg_n$$

has, in view of (2.3) and (2.7), the majorant

$$\left\{1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}\epsilon K 2^{-\nu(1-1/q)} = K'\epsilon 2^{-\nu(1-1/q)}.$$

Consequently, by (2.5)—on whose right-hand side the ranges $[0,\pi/n]$ and $[\frac{1}{2}\pi,\pi]$ are trivial—we have

$$\pi|s_n| \leqslant \sum_{2^\nu\leqslant n}|I_\nu|+\text{triv.} \leqslant K'\epsilon\sum_\nu 2^{-\nu(1-1/q)}+\text{triv.} \leqslant K''\epsilon.$$

This completes the proof.

3. The pth-power generalization of the Hardy-Littlewood refinements

For the reader acquainted with the details of Hardy and Littlewood (**1**), I give now a slightly more complete statement of the pth-power generalization of W. H. Young's convergence criterion. Given $p \geqslant 1$, we say that *f satisfies Y_p*, if, for all small t,

$$\phi(t) = O(1) \quad \text{and} \quad V_p(\phi;[t,2t]) = O(1). \tag{3.1}$$

Moreover, if (3.1) holds for all t in $[0,\pi]$, we say that *f satisfies Y_p^**, a condition which is only trivially more stringent than Y_p.

THEOREM 1. *If $\phi(t)$ satisfies Y_p, then the Fourier series is convergent whenever it is summable by any Cesàro mean. The necessary and sufficient condition for convergence (to sum s) is*

$$\int_0^t \phi(u)\,du = o(t). \tag{3.2}$$

If further $\phi(t)$ satisfies Y_p^*, then the series is summable $\{C, -(1/p)+\delta\}$ for every positive δ whenever it is summable by any Cesàro mean.

In order to prove Theorem 1, it is sufficient, by the argument of the last six lines of Hardy and Littlewood (**1**) § 3, 303, to prove

If $\phi(t)$ satisfies Y_p^*, then the series is bounded $\{C, -(1/p)+\delta\}$ for every positive δ. (3.3)

We may suppose again $\phi(t) = f(t)$. Writing

$$g_n^\gamma(t) = \int_0^t \Omega_1(t)\,dt,$$

the function $\Omega_1(t)$ being defined by the equation (2.4) of Hardy and Littlewood (**1**) § 3, 303, the Cesàro means of order $\gamma < 0$ of the Fourier series of f may be written

$$s_n^\gamma = \frac{1}{\pi}\int_{\pi/n}^{\pi} f\,dg_n^\gamma + \text{triv.} = \frac{1}{\pi}\sum_{2^{\nu+1}<n} I_\nu + \text{triv.}$$

where
$$I_\nu = \int_{2^\nu\pi/n}^{2^{\nu+1}\pi/n} f\,dg_n^\gamma.$$

Now, for sufficiently small δ, we choose

$$\gamma = -(1/p)+\delta \quad \text{and} \quad q = 1/(1+\gamma-\tfrac{1}{2}\delta) > 1,$$

and observe that
$$\frac{1}{p}+\frac{1}{q} = 1+\tfrac{1}{2}\delta > 1.$$

Since $q > 1$, we see, as in (**3**) § 12, 274, where g_n^γ is defined in a manner only trivially different from the present definition, that there exists a constant K such that

$$V_q^q(g_n^\gamma; [\alpha, \beta]) \leqslant K^q \sum_{n\alpha/\pi - 1 < m < n\beta/\pi + 1} (m+1)^{-q(\gamma+1)}$$

for any sub-interval $[\alpha, \beta]$ of $[0, \pi]$. Consequently,

$$V_q(g_n^\gamma; [2^\nu\pi/n, 2^{\nu+1}\pi/n]) \leqslant K\Big\{\sum_{2^\nu \leqslant m} (m+1)^{-q(\gamma+1)}\Big\}^{1/q}$$

$$\leqslant K 2^{-(\gamma+1-1/q)\nu} = K 2^{-\delta\nu/2}. \tag{3.4}$$

In view of Y_p^*, it therefore follows from the main inequality of (**3**), (10.9), 266, that

$$|I_\nu| \leqslant K' 2^{-\delta\nu/2},$$

so that

$$|s_n^\gamma| \leqslant K + K' \sum_\nu 2^{-\delta\nu/2} \leqslant K''.$$

This proves (3.3) for sufficiently small δ, and so for all positive δ, and thus completes the proof of Theorem 1.

4. Exponential forms of the criterion

We shall now show that (2.1) *may be replaced by*

$$f(t) = o(1) \quad \textit{and} \quad V_\Phi(f; [t, 2t]) = O(1), \tag{4.1}$$

where $V_\Phi(f; [\alpha, \beta])$ denotes the upper bound of the sum

$$\sum \Phi(|\Delta f|)$$

for all systems of non-overlapping sub-intervals Δ of $[\alpha, \beta]$, *provided that we choose for small* u

$$\Phi(u) = \exp(-u^{-c}) \quad \textit{where} \quad 0 < c < \tfrac{1}{2}.$$

The proof follows closely that of § 2 above, except that we use, in place of the results of (**3**), their generalizations in (**4**).

In the first place, for sufficiently small u_0, the inequalities $0 < u \leqslant u_0$ imply

$$\exp(-u^{-(c_1+c_2)}) \leqslant \exp(-u^{-c_1} - u^{-c_2}) \leqslant \exp(-u_0^{-c_2})\exp(-u^{-c_1}),$$

and from this it follows that by increasing c a little we may suppose O replaced by o in the second part of (4.1).

By the 'localization principle' it is therefore enough to show that the inequalities

$$|f(t)| \leqslant \epsilon \quad \textit{and} \quad V_\Phi(f; [t, 2t]) \leqslant \epsilon' \quad \textit{for all } t \textit{ of } (0, \pi) \tag{4.2}$$

imply $|s_n| < \epsilon''$, where ϵ'' denotes a function of ϵ and ϵ' independent of n which tends to zero as ϵ and ϵ' tend to zero simultaneously.

For this purpose we write $c = \frac{1}{2}\{1/(1+2\delta)\}$, where $\delta > 0$. Denoting by ϕ^* the inverse function† of Φ, we have for small u

$$\phi^*(u) = O\{\log(1/u)\}^{-2(1+2\delta)}$$

and therefore

$$\begin{aligned}\phi^*(\epsilon'/n) &= O(\log n + |\log \epsilon'|)^{-2(1+2\delta)} \\ &< K|\log \epsilon'|^{-\delta}(\log n)^{-2-3\delta} \\ &= \epsilon_1(\log n)^{-2-3\delta}, \text{ say.}\end{aligned} \tag{4.3}$$

Let us now, as in (**4**), § 7, choose for $\Psi(u)$ the function $u/|\log u|^{1+2\delta}$ when u is small and positive, and let $\psi(u)$ be the inverse function, which is of order $u|\log u|^{1+2\delta}$. For small B, we then have

$$\psi(B/n) < K'\psi(B)(\log n)^{1+2\delta}/n,$$

provided that $n > 1$; and hence, by (4.3), we obtain

$$\begin{aligned}\sum_n \phi^*(\epsilon'/n)\psi(B/n) &< \epsilon_1 \psi(B)\Big\{K + K' \sum_{n>1} 1/[n(\log n)^{1+\delta}]\Big\} \\ &< K''\epsilon_1 \psi(B).\end{aligned} \tag{4.4}$$

From the inequality of Theorem (5.1) (i) of (**4**), § 5, it follows that

$$\left|\int_t^{2t} f\, dg_n\right| \leqslant |f(t)|\,|g_n(2t)-g_n(t)| + K\epsilon_1 \psi(B), \tag{4.5}$$

where

$$B = V_\Psi(g_n; [t, 2t]),$$

provided that this number is sufficiently small. Moreover, since, by definition of V_Ψ, we then have

$$|g_n(2t)-g_n(t)| \leqslant \psi(B),$$

we may write (4.5) in the form

$$\left|\int_t^{2t} f\, dg_n\right| < K'(\epsilon+\epsilon_1)\psi(B). \tag{4.6}$$

Now when $t = 2^\nu\pi/n$, we have, by the argument of (**4**), § 7 (which generalizes the corresponding argument of (**3**) used above in §§ 2, 3 at this stage),

$$\begin{aligned}B \leqslant K \sum_{2^\nu \leqslant m \leqslant 2^{\nu+1}} \Psi(K'/(m+1)) &< K'' \sum_{2^\nu \leqslant m \leqslant 2^{\nu+1}} 1/m(\log m)^{1+2\delta} \\ &< K(\log 2^\nu)^{-(1+2\delta)} < K'\nu^{-(1+2\delta)},\end{aligned}$$

and therefore certainly $\psi(B) < K''\nu^{-(1+\delta)}$ for all large relevant ν.

† Not to be confused with the function $\phi(t)$ of the preceding paragraphs.

In view of (4.5), the absolute value of the integral

$$I_\nu = \int_{2^\nu\pi/n}^{2^{\nu+1}\pi/n} f\, dg_n$$

thus has the majorant $K(\epsilon+\epsilon_1)\nu^{-(1+\delta)}$ for all relevant $\nu \geqslant \nu_0$ independent of n. Hence, finally,

$$\begin{aligned} \pi|s_n| &\leqslant \sum_{\nu \geqslant \nu_0} |I_\nu| + \text{triv.} \\ &\leqslant K(\epsilon+\epsilon_1) \sum_\nu \nu^{-(1+\delta)} + \text{triv.} \\ &\leqslant \epsilon'', \end{aligned}$$

and this completes the proof.

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ON FOURIER SERIES SATISFYING MIXED CONDITIONS

By E. H. LINFOOT (*Bristol*) and W. M. SHEPHERD (*Bristol*)

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1. THE series $\sum_{n=0}^{\infty} A_n \cos n\theta$, $\sum_{n=1}^{\infty} A_n \sin n\theta$, where

$$A_n = A_{n,m} = \begin{cases} (-1)^{k+\nu+1} \dfrac{[k][\nu](2k+1)}{2\nu+2k+2} & (m = 2k+1;\ n = 2\nu+1) \\ (-1)^{k+\nu+1} \dfrac{[k][\nu](2k+1)}{2\nu-2k-1} & (m = 2k+1;\ n = 2\nu) \\ (-1)^{k+\nu} \dfrac{[k][\nu]2k}{2\nu-2k+1} & (m = 2k;\ n = 2\nu+1) \\ (-1)^{k+\nu} \dfrac{[k][\nu]2k}{2\nu+2k} & (m = 2k;\ n = 2\nu) \end{cases} \tag{1}$$

and $[n]$ denotes $\dfrac{1.3 \ldots (2n-1)}{2.4 \ldots 2n}$, were shown by W. M. Shepherd* to have the property

$$\left.\begin{aligned} \sum_{n=0}^{\infty} A_n \cos n\theta &= \cos m\theta && (|\theta| < \tfrac{1}{2}\pi) \\ \sum_{n=1}^{\infty} A_n \sin n\theta &= -\sin m\theta && (\tfrac{1}{2}\pi < |\theta| < \pi) \end{aligned}\right\}. \tag{2}$$

The main object of the present note is to discuss to what extent this solution of the equations

$$\left.\begin{aligned} \sum_{n=0}^{\infty} \alpha_n \cos n\theta &= \cos m\theta && (|\theta| < \tfrac{1}{2}\pi) \\ \sum_{n=1}^{\infty} \alpha_n \sin n\theta &= -\sin m\theta && (\tfrac{1}{2}\pi < |\theta| < \pi) \end{aligned}\right\} \tag{3}$$

is unique. Strict uniqueness is not to be expected, since formal differentiation of (2) leads to the result

$$\left.\begin{aligned} \sum_{n=1}^{\infty} nA_n \cos n\theta &= -m \cos m\theta && (\tfrac{1}{2}\pi < |\theta| < \pi) \\ \sum_{n=1}^{\infty} nA_n \sin n\theta &= m \sin m\theta && (|\theta| < \tfrac{1}{2}\pi) \end{aligned}\right\}, \tag{4}$$

* *Proc. London Math. Soc.* (2) 43 (1937), 366–75.

from which, on replacing θ by $\pi-\theta$, we obtain the formal second solution

$$\left.\begin{aligned}\sum_{n=1}^{\infty} b_n \cos n\theta &= \cos m\theta \qquad (|\theta| < \tfrac{1}{2}\pi)\\ \sum_{n=1}^{\infty} b_n \sin n\theta &= -\sin m\theta \quad (\tfrac{1}{2}\pi < |\theta| < \pi),\end{aligned}\right\} \tag{5}$$

where
$$b_n = (-1)^{m+n+1}\frac{n}{m}A_n.$$

We shall prove that (5) actually is a second solution and that the most general solution of (3) by a pair of conjugate Fourier series is a linear combination of the two solutions (2) and (5). More precisely, we prove

THEOREM 1. *For every given α_0, there is just one pair of conjugate Fourier series $\sum_{n=0}^{\infty} \alpha_n \cos n\theta$, $\sum_{n=1}^{\infty} \alpha_n \sin n\theta$ satisfying* (3), *namely that given by*

$$\alpha_n = \frac{\alpha_0}{A_0}A_n + \left(1-\frac{\alpha_0}{A_0}\right)b_n \quad (n \geqslant 1). \tag{6}$$

In the course of the analysis we obtain, by an elementary application of complex-function theory, a second proof of the equations (2) in a more complete form, namely one which gives the sums of the series over the whole range $(-\pi, \pi)$. The final result is

THEOREM 2. *Let r' denote the greatest integer not exceeding $\frac{1}{2}r$. Then*

$$\sum_{n=}^{\infty} A_n \cos n\theta = \begin{cases} \cos m\theta \quad (|\theta| < \tfrac{1}{2}\pi), & (7)\\ \cos m\theta - 2\sqrt{|2\cos\theta|}\displaystyle\sum_{r=0}^{(m+1)'-1}(-1)^r[r]\sin(2r-m+\tfrac{1}{2})\theta \\ \qquad\qquad (\tfrac{1}{2}\pi < |\theta| < \pi); & (8)\end{cases}$$

$$\sum_{n=1}^{\infty} A_n \sin n\theta = \begin{cases} -\sin m\theta - 2\sqrt{(2\cos\theta)}\displaystyle\sum_{r=0}^{(m+1)'-1}(-1)^r[r]\sin(2r-m+\tfrac{1}{2})\theta \\ \qquad\qquad (|\theta| < \tfrac{1}{2}\pi), & (9)\\ -\sin m\theta \quad (\tfrac{1}{2}\pi < |\theta| < \pi). & (10)\end{cases}$$

2. Proof of Theorem 2

LEMMA 1. *For $k \geqslant 0$*

$$I_k = \int_0^{\infty}\left\{\frac{x}{\sqrt{(1+x^2)}} - \sum_{m=0}^{k}(-1)^m[m]x^{-2m}\right\}x^{2k}\,dx = \frac{(-1)^{k+1}}{(2k+2)[k+1]}.$$

Proof. By moving the path of integration round on to the y-axis in the complex plane, we easily obtain

$$I_k = (-1)^{k+1}\sum_{m=0}^{\infty}\frac{[m]}{2k+2m+2}-(-1)^k i\sum_{m=0}^{\infty}\frac{[m]}{2k-2m+1}$$
$$= (-1)^{k+1}\frac{1}{(2k+2)[k+1]}$$

by equations (18), (20), loc. cit. supra.

LEMMA 2. *For $k \geqslant 0$,*

$$J_k = \int\limits_0^1 \frac{u^k\,du}{\sqrt{(u+e^{-2i\theta})}}$$
$$= \frac{(-1)^k}{[k+1](2k+2)}\left\{-2e^{-(2k+1)i\theta}+2e^{-(2k+\frac{1}{2})i\theta}\sqrt{(2\cos\theta)}\sum_{r=0}^{k}(-1)^r[r]e^{2ir\theta}\right\},$$

where $\sqrt{(u+e^{-2i\theta})}$ denotes the branch which is $e^{-i\theta}$ when $u = 0$.

Proof.

(i) $$J_0 = \int\limits_0^1 \frac{du}{\sqrt{(u+e^{-2i\theta})}} = [2\sqrt{(u+e^{-2i\theta})}]_0^1 = 2e^{-\frac{1}{2}i\theta}\sqrt{(2\cos\theta)}-2e^{-i\theta};$$

(ii) for $k \geqslant 1$,

$$J_k = \int\limits_0^1 \frac{u^k\,du}{\sqrt{(u+e^{-2i\theta})}} = [2u^k\sqrt{(u+e^{-2i\theta})}]_0^1-2k\int\limits_0^1\sqrt{(u+e^{-2i\theta})}u^{k-1}\,du$$
$$= 2\sqrt{(1+e^{-2i\theta})}-2kJ_k-2ke^{-2i\theta}J_{k-1},$$
$$J_k = \frac{2e^{-\frac{1}{2}i\theta}\sqrt{(2\cos\theta)}}{2k+1}-\frac{2k}{2k+1}e^{-2i\theta}J_{k-1}.$$

The lemma now follows by (i) and successive applications of (ii).

Summation of the series (2). We give the argument only in the case where $m = 2k+1$ is odd; the proof for even m is substantially the same.

Consider the function

$$\sum_{n=0}^{\infty}A_n z^{-n} = \sum_{r=0}^{\infty}(-1)^{r+k+1}[r][k+1]\frac{2k+2}{2r-2k-1}z^{-2r}-$$
$$-\sum_{r=0}^{\infty}(-1)^{r+k+1}[r][k+1]\frac{2k+2}{2k+2r+2}z^{-2r-1}$$

$$= (-1)^{k+1}[k+1](2k+2)\Big\{z^{-2k-1}\sum_{r=0}^{\infty}(-1)^r\frac{[r]}{2r-2k-1}z^{-(2r-2k-1)}+$$
$$+z^{2k+1}\sum_{r=0}^{\infty}(-1)^r\frac{[r]}{2k+2r+2}z^{-(2r+2k+2)}\Big\}$$
$$= (-1)^{k}[k+1](2k+2)\Big\{z^{-2k-1}\int_0^z\sum_{r=0}^{\infty}(-1)^r[r]\zeta^{-2r+2k}\,d\zeta+$$
$$+z^{-2k-1}\int_\infty^z\sum_{r=k+1}^{\infty}(-1)^r[r]\zeta^{-2r+2k}\,d\zeta+$$
$$+z^{2k+1}\int_\infty^z\sum_{r=0}^{\infty}(-1)^r[r]\zeta^{-2r-2k-3}\,d\zeta\Big\}$$
$$= (-1)^{k}[k+1](2k+2)\Big\{z^{-2k-1}\int_0^z\sum_{r=0}^{k}(-1)^r[r]\zeta^{-2r}\zeta^{2k}\,d\zeta+$$
$$+z^{-2k-1}\int_\infty^z\Big[\Big(1+\frac{1}{\zeta^2}\Big)^{-\frac{1}{2}}-\sum_{r=0}^{k}(-1)^r[r]\zeta^{-2r}\Big]\zeta^{2k}\,d\zeta+$$
$$+z^{2k+1}\int_\infty^z\Big(1+\frac{1}{\zeta^2}\Big)^{-\frac{1}{2}}\zeta^{-2k-3}\,d\zeta\Big\}. \quad (11)$$

Here the paths of integration are all parts of radii through the origin; the branch of $\{1+(1/\zeta^2)\}^{-\frac{1}{2}}$ is that which tends to $+1$ at ∞. In the last integral we make the transformation $\zeta = 1/\omega$; it then becomes

$$-\int_0^{1/z}\Big(1+\frac{1}{\omega^2}\Big)^{-\frac{1}{2}}\omega^{2k}\,d\omega$$

(the branch of $\{1+(1/\omega^2)\}^{-\frac{1}{2}}$ being that which is asymptotic to $+\omega$ at the origin). We now have

$$\sum_{n=0}^{\infty}A_n z^{-n} = (-1)^{k+1}[k+1](2k+2)z^{-2k-1}\times$$
$$\times\Big(\int_0^z+\int_z^\infty\Big)\Big\{\Big(1+\frac{1}{\zeta^2}\Big)^{-\frac{1}{2}}-\sum_{r=0}^{k}(-1)^r[r]\zeta^{-2r}\Big\}\zeta^{2k}\,d\zeta+$$
$$+(-1)^k[k+1](2k+2)\Big\{z^{-2k-1}\int_0^z\Big(1+\frac{1}{\zeta^2}\Big)^{-\frac{1}{2}}\zeta^{2k}\,d\zeta-$$
$$-z^{2k+1}\int_0^{1/z}\Big(1+\frac{1}{\omega^2}\Big)^{-\frac{1}{2}}\omega^{2k}\,d\omega\Big\}, \quad (12)$$

where $\{1+(1/\zeta^2)\}^{-\frac{1}{2}}$ denotes the branch which tends to $+1$ at ∞, $\{1+(1/\omega^2)\}^{-\frac{1}{2}}$ the branch which is asymptotic to $+\omega$ at 0.

Now suppose $z = e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$. Then $1/z = \bar{z}$ and the last term is a pure imaginary, since, along the radius through $\bar{z}$, the branch of $\{1+(1/\zeta^2)\}^{-\frac{1}{2}}$ which is $+1$ at ∞ is asymptotic to $+\zeta$ at 0. Also the first term on the right of (12) is equal to

$$-z^{-2k-1}(-1)^k[k+1](2k+2)\int\limits_0^\infty \left\{\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}} - \right.$$
$$\left. -\sum_{m=0}^{k}(-1)^m[m]x^{-2m}\right\}x^{2k}\,dx = z^{-2k-1}$$

by Lemma 1. Thus for $z = e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$,

$$\sum_{n=0}^{\infty} A_n z^{-n} = z^{-2k-1}+(-1)^k[k+1](2k+2)\left\{z^{-2k-1}\int\limits_0^z \left(1+\frac{1}{\zeta^2}\right)^{-\frac{1}{2}}\zeta^{2k}\,d\zeta - \right.$$
$$\left. -z^{2k+1}\int\limits_0^{\bar{z}}\left(1+\frac{1}{\omega^2}\right)^{-\frac{1}{2}}\omega^{2k}\,d\omega\right\}.$$

In the first integral put $\zeta = u^{\frac{1}{2}}z = u^{\frac{1}{2}}e^{i\theta}$ $(0 \leqslant u^{\frac{1}{2}} \leqslant 1)$; in the second put $\omega = u^{\frac{1}{2}}\bar{z} = u^{\frac{1}{2}}e^{-i\theta}$. Then

$$\sum_{n=0}^{\infty} A_n z^{-n} = z^{-2k-1}+i(-1)^k[k+1](2k+2)\Im\left(\int\limits_0^1 \frac{u^k\,du}{\sqrt{(u+e^{-2i\theta})}}\right).$$

Equating real and imaginary parts and using Lemma 2, we obtain equations (7) and (9) of Theorem 2.

If on the other hand $z = e^{i\theta}$, $\frac{1}{2}\pi < |\theta| < \pi$, then the last term of (12) is purely real, since now the branch of $\{1+(1/\zeta)\}^{-\frac{1}{2}}$ which is $+1$ at ∞ is asymptotic to $-\zeta$ at 0, while the first term is now equal to $-z^{-2k-1}$. We therefore obtain

$$\sum_{n=0}^{\infty} A_n z^{-n} = -z^{-2k-1}+(-1)^{k+1}[k+1](2k+2)\Re\left(\int\limits_0^1 \frac{u^k\,du}{\sqrt{(u+e^{-2i\theta})}}\right),$$

and on equating real and imaginary parts this yields (8), (10) of Theorem 2.

3. Proof of Theorem 1

(i) The series given by (6) actually have the required properties. To prove this it is enough to show that the series (5) are Fourier series and that the equations (5) hold in the sense of ordinary con-

vergence. Now by Theorem 2 the sums of the series (2) are absolutely continuous functions over $(-\pi, \pi)$. It follows that their Fourier series, namely the series (2), give on term-by-term differentiation the Fourier series of the derived functions. These derivatives are, by Theorem 2, themselves differentiable except at the points $\pm\frac{1}{2}\pi$. Hence their Fourier series converge, except at these points, to the derived functions as sums. This establishes the equations (4), and hence also (5), in the sense of ordinary convergence.

(ii) To prove the uniqueness, we make use of the following theorem, which is an easy deduction from results of Volterra* and Fatou.†

THEOREM A. *Suppose that $g(\mu)$ is continuous and bounded in the range $-\alpha < \mu < \alpha$, and let*

$$H(\mu) = \begin{cases} g(\mu) \quad (|\mu| < \alpha), & (13) \\ -\dfrac{1}{2\pi}\displaystyle\int_{-\alpha}^{\alpha} g(\lambda)\sqrt{\left(\frac{\cos\lambda - \cos\alpha}{\cos\alpha - \cos\mu}\right)}|\operatorname{cosec}\tfrac{1}{2}(\lambda-\mu)|\,d\lambda \\ \qquad\qquad (|\alpha| < |\mu| < \pi). & (14) \end{cases}$$

Then $H(\mu)$ is integrable over $(-\pi, \pi)$ and possesses a conjugate function

$$\bar{H}(\mu) = \begin{cases} \dfrac{1}{2\pi} P\displaystyle\int_{-\alpha}^{\alpha} g(\lambda)\sqrt{\left(\frac{\cos\lambda - \cos\alpha}{\cos\mu - \cos\alpha}\right)}\operatorname{cosec}\tfrac{1}{2}(\lambda-\mu)\,d\lambda \quad (|\mu| < |\alpha|), \\ & (15) \\ 0 \quad (|\alpha| < |\mu| < \pi). & (16) \end{cases}$$

Further,
$$\int_{-\pi}^{\pi} H(\mu)\,d\mu = 0, \qquad \int_{-\pi}^{\pi} \bar{H}(\mu)\,d\mu = 0. \qquad (17), (18)$$

Finally, there is is only one function $H(\mu)$ satisfying (13), (16), (17), *and this satisfies also* (14), (15), (18).

It is readily verified that the function $h_0(\mu)$, defined by

$$h_0(\mu) = \begin{cases} 0 \quad (|\mu| < \alpha), & (19) \\ \dfrac{|\sin\frac{1}{2}\mu|}{\pi\sqrt{\{2(\cos\alpha - \cos\mu)\}}} \quad (\alpha < |\mu| < \pi) & (20) \end{cases}$$

possesses a conjugate function

$$\overline{h_0(\mu)} = \frac{1}{2\pi} P\int_{-\pi}^{\pi} h(\theta)\cot\tfrac{1}{2}(\mu-\theta)\,d\theta = \begin{cases} \dfrac{-\sin\frac{1}{2}\mu}{\pi\sqrt{\{2(\cos\mu - \cos\alpha)\}}} \\ \qquad\qquad (|\mu| < \alpha), & (21) \\ 0 \quad (\alpha < |\mu| < \pi), & (22) \end{cases}$$

* V. Volterra, *Annali di Mat.* (2) 11 (1882), 1–55.

† P. Fatou, *Acta Math.* 30 (1906), 335–400.

and that
$$\int_{-\pi}^{\pi} h_0(\mu)\,d\mu = 1, \qquad \int_{-\pi}^{\pi} \overline{h_0(\mu)}\,d\mu = 0. \tag{23), (24}$$

It follows from Theorem A that two different functions cannot both satisfy (19), (22), (23); hence $h_0(\mu)$ as defined by (19), (20) is the only solution of these equations. This gives

THEOREM B. *If in Theorem A condition* (17) *is replaced by*

$$\int_{-\pi}^{\pi} H(\mu)\,d\mu = c, \tag{17'}$$

then for any given value of c *there is just one function satisfying conditions* (13), (16), (17′), *namely that obtained by adding* $ch_0(\mu)$ *to the unique solution of* (13), (16), (17). *The new solution still satisfies* (18).

To apply Theorem B, we observe that it follows from equations (3) that the function

$$H_1(\mu) \sim \cos m\mu + \sum_{m=0}^{\infty} \alpha_n \cos n\mu \quad (-\pi < \mu < \pi)$$

satisfies the hypotheses of Theorem B with

$$\alpha = \tfrac{1}{2}\pi, \qquad c = 2\pi\alpha_0, \qquad g(\mu) = 2\cos m\mu.$$

Hence, given α_0, $H_1(\mu)$ is unique, and so therefore is its Fourier series. Thus Theorem 1 is established.

A NOTE ON GEGENBAUER POLYNOMIALS

By G. N. WATSON (*Birmingham*)

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THERE is a well-known formula, due to Christoffel,* which expresses 'in finite terms' the second solution $Q_n(z)$ of Legendre's equation for functions of integral order n; the formula in question is

$$Q_n(z) = \tfrac{1}{2}P_n(z)\log\frac{z+1}{z-1} - \sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]} \frac{2n-4m-1}{(n-m)(2m+1)} P_{n-2m-1}(z). \tag{1}$$

where $[x]$ denotes the greatest integer contained in x. Other expressions for the sum on the right have been obtained by Schläfli,† Hermite,‡ and Hobson.§

In this note I obtain a straightforward generalization of (1) for the second solution of the differential equation satisfied by Gegenbauer's polynomial $C_n^\nu(z)$. From my result I deduce an interesting formula involving a generalized hypergeometric function of type ${}_6F_5$ and argument unity. I have not been successful in obtaining a result of the same simple character for the more general polynomials of Jacobi; it seems essential to my analysis that it should have to deal with hypergeometric functions which are susceptible to a quadratic transformation.

It will be remembered that the Gegenbauer polynomials are defined by the expansion

$$\frac{1}{(1-2hz+h^2)^\nu} = \sum_{n=0}^{\infty} h^n C_n^\nu(z);$$

it is well known that $C_n^\nu(z)$ is a solution of the differential equation

$$(z^2-1)\frac{d^2y}{dz^2} + (2\nu+1)z\frac{dy}{dz} - n(2\nu+n)y = 0, \tag{2}$$

and that it is expressible in the forms

$$C_n^\nu(z) = \frac{\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)}\,{}_2F_1(-n, 2\nu+n;\, \nu+\tfrac{1}{2};\, \tfrac{1}{2}-\tfrac{1}{2}z), \tag{3}$$

$$C_n^\nu(z) = \frac{2^n\Gamma(\nu+n)z^n}{n!\,\Gamma(\nu)}\,{}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n;\, 1-\nu-n;\, z^{-2}). \tag{4}$$

* E. B. Christoffel, *Journal für Math.* 55 (1858), 61–82.

† L. Schläfli, *Ueber die zwei Heine'schen Kugelfunktionen* ... (Bern, 1881), 61.

‡ C. Hermite, *Jornal de Ciencias mat.* 6 (1885), 81–4.

§ E. W. Hobson, *Spherical and Ellipsoidal Harmonics* (Cambridge, 1931), 54.

The reader should have no difficulty in verifying the following recurrence formulae (with $n \geqslant 1$):

$$(n+1)C_{n+1}^{\nu}(z)-2(n+\nu)zC_n^{\nu}(z)+(n+2\nu-1)C_{n-1}^{\nu}(z) = 0, \tag{5}$$

$$z\frac{dC_n^{\nu}(z)}{dz}-\frac{dC_{n-1}^{\nu}(z)}{dz} = nC_n^{\nu}(z), \tag{6}$$

$$\frac{dC_{n+1}^{\nu}(z)}{dz}-\frac{dC_{n-1}^{\nu}(z)}{dz} = 2(n+\nu)C_n^{\nu}(z); \tag{7}$$

when $n = 0$, (5) is to be replaced by

$$C_1^{\nu}(z)-2\nu zC_0^{\nu}(z) = 0; \tag{5a}$$

and, in like manner, the other recurrence formulae are made valid for $n = 0$ merely by the suppression of functions of negative degree.

Now it is easy to prove that, for $|z|$ sufficiently large and 2ν not zero or a negative integer, the two functions

$$z^{-2\nu-n}{}_2F_1(\nu+\tfrac{1}{2}n,\, \nu+\tfrac{1}{2}n+\tfrac{1}{2};\, \nu+n+1;\, z^{-2}),$$

$$(z-1)^{-2\nu-n}{}_2F_1\left(2\nu+n,\, \nu+n+\tfrac{1}{2};\, 2\nu+2n+1;\, \frac{2}{1-z}\right)$$

are both solutions of (2); and, by a comparison of their behaviour with that of $C_n^{\nu}(z)$ when $|z|$ is large, we see that the two functions are different from $C_n^{\nu}(z)$ and that they are identically equal throughout the z-plane cut along the real axis from $z = -\infty$ to $z = +1$. We shall adopt as the standard second solution of (2) the function $D_n^{\nu}(z)$ defined by the formulae*

$$\begin{aligned} D_n^{\nu}(z) &= \frac{\Gamma(\nu)\Gamma(2\nu+n)z^{-2\nu-n}}{2^{n+1}\Gamma(\nu+n+1)}\,{}_2F_1(\nu+\tfrac{1}{2}n,\, \nu+\tfrac{1}{2}n+\tfrac{1}{2};\, \nu+n+1;\, z^{-2}) \\ &= \frac{\Gamma(\nu)\Gamma(2\nu+n)(z-1)^{-2\nu-n}}{2^{n+1}\Gamma(\nu+n+1)}\times \\ &\qquad \times {}_2F_1\left(2\nu+n,\, \nu+n+\tfrac{1}{2};\, 2\nu+2n+1;\, \frac{2}{1-z}\right); \end{aligned} \tag{8}$$

in particular, it is easy to prove that, when $R(\nu) > 0$,

$$D_0^{\nu}(z) = \Gamma(2\nu)\int_z^{\infty}\frac{dt}{(t^2-1)^{\nu+\frac{1}{2}}}, \qquad D_1^{\nu}(z) = z\Gamma(2\nu)\int_z^{\infty}\frac{dt}{t^2(t^2-1)^{\nu+\frac{1}{2}}}.$$

It is easily verified that $D_n^{\nu}(z)$ satisfies the same recurrence formulae

* This is the definition adopted by L. Gegenbauer, *Sitz. der K. Akad. der Wiss. zu Wien*, 75 (1877), 891–905. The definition presupposes that $2\nu-1$ is not a negative integer.

as $C_n^\nu(z)$ for general values of n; in particular, it satisfies (5), but, when $n = 0$, the formula which corresponds to (5a) is

$$D_1^\nu(z) - 2\nu z D_0^\nu(z) + \frac{\Gamma(2\nu)}{(z^2-1)^{\nu-\frac{1}{2}}} = 0. \tag{5b}$$

Now it is an immediate consequence of (5) that the functions

$$u_m \equiv \frac{m!\,\Gamma(\nu)C_m^\nu(z)}{2^m\Gamma(\nu+m)} \quad \text{and} \quad u_m \equiv \frac{m!\,\Gamma(\nu)D_m^\nu(z)}{2^m\Gamma(\nu+m)}$$

are solutions of the difference equation

$$u_{m+1} - zu_m + a_m u_{m-1} = 0 \quad (m = 1, 2, 3, \ldots), \tag{9}$$

where
$$a_m = \frac{m(m+2\nu-1)}{4(\nu+m)(\nu+m-1)};$$

for $m = 0$ we have the two special equations

$$u_1 - zu_0 = 0, \tag{9a}$$

$$u_1 - zu_0 + \frac{\Gamma(2\nu)}{2\nu(z^2-1)^{\nu-\frac{1}{2}}} = 0, \tag{9b}$$

satisfied by the appropriate values of u_1 and u_0 in the respective cases.

Let us now write down equation (9) for $m = 1, 2, 3, \ldots, n-1$ with the function of the second kind inserted, and from these equations combined with (9b) let us eliminate $D_1^\nu(z)$, $D_2^\nu(z), \ldots, D_{n-1}^\nu(z)$. The result of the elimination is

$$\frac{n!\,\Gamma(\nu)D_n^\nu(z)}{2^n\Gamma(\nu+n)} = \Delta_n(z)D_0^\nu(z) - \frac{\Gamma(2\nu)\Delta_n'(z)}{2\nu(z^2-1)^{\nu-\frac{1}{2}}},$$

where

$$\Delta_n(z) = \begin{vmatrix} z & a_{n-1} & 0 & 0 & . & . & . & 0 & 0 \\ 1 & z & a_{n-2} & 0 & . & . & . & 0 & 0 \\ 0 & 1 & z & a_{n-3} & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & z & a_1 \\ 0 & 0 & 0 & 0 & . & . & . & 1 & z \end{vmatrix}$$

and $\Delta_n'(z)$ is the determinant obtained by suppressing the last row and the last column of the determinant $\Delta_n(z)$; for the result to hold when $n = 1$, we adopt the convention that $\Delta_1'(z) = 1$.

In precisely the same manner from (9) and (9a) it follows that

$$\frac{n!\,\Gamma(\nu)C_n^\nu(z)}{2^n\Gamma(\nu+n)} = \Delta_n(z)C_0^\nu(z) = \Delta_n(z),$$

and hence we have

$$\frac{D_n^\nu(z)}{\Gamma(2\nu)} = \frac{C_n^\nu(z)D_0^\nu(z)}{\Gamma(2\nu)} - \frac{2^{n-1}\Gamma(\nu+n)\Delta_n'(z)}{n!\,\Gamma(\nu+1)(z^2-1)^{\nu-\frac{1}{2}}}.$$

This result can evidently be written in the form

$$\frac{D_n^\nu(z)}{\Gamma(2\nu)} = \frac{C_n^\nu(z)D_0^\nu(z)}{\Gamma(2\nu)} - \frac{W_{n-1}(z)}{(z^2-1)^{\nu-\frac{1}{2}}}, \tag{10}$$

where $W_{n-1}(z)$ is obviously a polynomial in z of degree $n-1$; it contains only odd powers or only even powers of z according as n is even or odd; and $W_{-1}(z) = 0$. The result holds when 2ν is not a negative integer.

We now have to face the task of expressing $W_{n-1}(z)$ in some form simpler than the determinant by which it has been defined; it is evidently permissible to assume an expansion of the form

$$W_{n-1}(z) = \sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]} \lambda_m C_{n-2m-1}^\nu(z), \tag{11}$$

where the coefficients λ_m are constants as yet undetermined. To determine their values, we observe that, as a consequence of (10),

$$\left\{(z^2-1)\frac{d^2}{dz^2} + (2\nu+1)z\frac{d}{dz} - n(2\nu+n)\right\}\frac{W_{n-1}(z)}{(z^2-1)^{\nu-\frac{1}{2}}}$$
$$= \left\{(z^2-1)\frac{d^2}{dz^2} + (2\nu+1)z\frac{d}{dz} - n(2\nu+n)\right\}\frac{C_n^\nu(z)D_0^\nu(z)}{\Gamma(2\nu)}$$
$$= \frac{2(z^2-1)}{\Gamma(2\nu)}\frac{dC_n^\nu(z)}{dz}\frac{dD_0^\nu(z)}{dz} = -\frac{2}{(z^2-1)^{\nu-\frac{1}{2}}}\frac{dC_n^\nu(z)}{dz},$$

and hence that

$$\left\{(z^2-1)\frac{d^2}{dz^2} - (2\nu-3)z\frac{d}{dz} - (n+1)(2\nu+n-1)\right\}W_{n-1}(z) = -2\frac{dC_n^\nu(z)}{dz}.$$

When we substitute the series (11) for $W_{n-1}(z)$ and use the differential equation satisfied by $C_{n-2m-1}^\nu(z)$, we see at once that

$$\frac{dC_n^\nu(z)}{dz} = \sum_{m=0}^{\infty}\lambda_m\left\{(2\nu-1)z\frac{dC_{n-2m-1}^\nu(z)}{dz} - \right.$$
$$\left. -\{2(m+1)(\nu+n-m)-(n+1)\}C_{n-2m-1}^\nu(z)\right\}.$$

Now, from the recurrence formulae (6) and (7), it is clear that

$$(2\nu-1)z\frac{dC^{\nu}_{n-2m-1}(z)}{dz}-\{2(m+1)(\nu+n-m)-(n+1)\}C^{\nu}_{n-2m-1}(z)$$

$$=(2\nu-1)\frac{dC^{\nu}_{n-2m-2}(z)}{dz}+2(n-m)(\nu+m)C^{\nu}_{n-2m-1}(z)$$

$$=(2\nu-1)\frac{dC^{\nu}_{n-2m-2}(z)}{dz}+\frac{(n-m)(\nu+m)}{\nu+n-2m-1}\left\{\frac{dC^{\nu}_{n-2m}(z)}{dz}-\frac{dC^{\nu}_{n-2m-2}(z)}{dz}\right\}$$

$$=\frac{(n-m)(\nu+m)}{\nu+n-2m-1}\frac{dC^{\nu}_{n-2m}(z)}{dz}-\frac{(m+1-\nu)(2\nu+n-m-1)}{\nu+n-2m-1}\frac{dC^{\nu}_{n-2m-2}(z)}{dz},$$

and so

$$\frac{dC^{\nu}_{n}(z)}{dz}=\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}\lambda_m\left\{\frac{(n-m)(\nu+m)}{\nu+n-2m-1}\frac{dC^{\nu}_{n-2m}(z)}{dz}-\right.$$

$$\left.-\frac{(m+1-\nu)(2\nu+n-m-1)}{\nu+n-2m-1}\frac{dC^{\nu}_{n-2m-2}(z)}{dz}\right\}.$$

It is, however, obvious that necessary and sufficient conditions for the sum

$$\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}\mu_m\frac{dC^{\nu}_{n-2m}(z)}{dz}$$

to be identically zero are that all the coefficients μ_m should vanish. Hence we have

$$\lambda_0=\frac{\nu+n-1}{\nu n},$$

$$\frac{(n-m)(\nu+m)}{\nu+n-2m-1}\lambda_m=\frac{(m-\nu)(2\nu+n-m)}{\nu+n-2m+1}\lambda_{m-1},$$

where m assumes the values 1, 2, 3,..., $[\frac{1}{2}n-\frac{1}{2}]$. We thus get, for $m=0, 1, 2,..., [\frac{1}{2}n-\frac{1}{2}]$,

$$\lambda_m=(\nu+n-2m-1)\frac{(n-m-1)!\,\Gamma(\nu)\Gamma(1-\nu+m)\Gamma(2\nu+n)}{n!\,\Gamma(\nu+m+1)\Gamma(1-\nu)\Gamma(2\nu+n-m)}.$$

If, for brevity, we now write

$$\alpha(\alpha+1)(\alpha+2)...(\alpha+m-1)\equiv(\alpha)_m,$$

and adopt the convention that empty products are to be interpreted to mean unity, we have

$$\lambda_m=(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}.$$

We have consequently obtained the required generalization of Christoffel's formula in the form

$$\frac{D_n^\nu(z)}{\Gamma(2\nu)} = \frac{C_n^\nu(z)D_0^\nu(z)}{\Gamma(2\nu)} -$$
$$-\frac{1}{(z^2-1)^{\nu-\frac{1}{2}}}\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}C_{n-2m-1}^\nu(z), \quad (12)$$

subject to the proviso that $2\nu-1$ must not be a negative integer.

When $R(\nu) > 0$, we may rewrite (12) in the modified form

$$\frac{D_n^\nu(z)}{\Gamma(2\nu)} = C_n^\nu(z)\int\limits_z^\infty \frac{dt}{(t^2-1)^{\nu+\frac{1}{2}}} -$$
$$-\frac{1}{(z^2-1)^{\nu-\frac{1}{2}}}\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}C_{n-2m-1}^\nu(z), \quad (13)$$

and this immediately reduces to Christoffel's formula when $\nu = \frac{1}{2}$.

We now obtain a limiting form of this result by making $z \to 1+0$. For values of z such that $|z-1| < 2$, the analytic continuation of (8) gives*

$$\frac{D_n^\nu(z)}{\Gamma(2\nu)} = \frac{\Gamma(2\nu+n)\Gamma(\frac{1}{2}-\nu)}{2^{2\nu}n!\,\Gamma(\nu+\frac{1}{2})}\,{}_2F_1(2\nu+n, -n; \nu+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}z)+$$
$$+\frac{(\frac{1}{2}z-\frac{1}{2})^{\frac{1}{2}-\nu}}{2^{2\nu}(\nu-\frac{1}{2})}\,{}_2F_1(\nu+n+\tfrac{1}{2}, \tfrac{1}{2}-\nu-n; \tfrac{3}{2}-\nu; \tfrac{1}{2}-\tfrac{1}{2}z),$$

with the additional restriction that $\nu-\frac{1}{2}$ must not be an integer. In particular, we have

$$\frac{D_0^\nu(z)}{\Gamma(2\nu)} = \frac{\Gamma(\nu)\Gamma(\frac{1}{2}-\nu)}{2\Gamma(\frac{1}{2})} + \frac{\frac{1}{2}(z-\frac{1}{2})^{\frac{1}{2}-\nu}}{2^{2\nu}(\nu-\frac{1}{2})}\,{}_2F_1(\nu+\tfrac{1}{2}, \tfrac{1}{2}-\nu; \tfrac{3}{2}-\nu; \tfrac{1}{2}-\tfrac{1}{2}z).$$

If we now multiply through (12) by $(z^2-1)^{\nu-\frac{1}{2}}$ and then make $z \to 1+0$, we see that

$$\frac{1}{2\nu-1} = \frac{C_n^\nu(1)}{2\nu-1} - \sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}C_{n-2m-1}^\nu(1),$$

and hence

$$\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m(2\nu-1)_{n-2m}}{(n-m)_{m+1}(\nu)_{m+1}(n-2m-1)!} = \frac{\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)} - 1. \quad (14)$$

* Cf. E. W. Barnes, *Proc. London Math. Soc.* (2), 6 (1908), 141–77.

When we adopt the notation of generalized hypergeometric functions, (14) assumes the form

$$ {}_6F_5\left[\begin{matrix}1,\ \tfrac{1}{2}(3-\nu-n),\ 1-\nu,\ 1-2\nu-n,\ \tfrac{1}{2}(1-n),\ \tfrac{1}{2}(2-n);\\ \tfrac{1}{2}(1-\nu-n),\ 1-n,\ 1+\nu,\ \tfrac{1}{2}(3-n-2\nu),\ \tfrac{1}{2}(2-n-2\nu)\end{matrix}\right] $$

$$ = \frac{\nu(2\nu+n-1)}{(2\nu-1)(\nu+n-1)} - \frac{n!\,\nu\,\Gamma(2\nu-1)}{(\nu+n-1)\Gamma(2\nu+n-1)}. \quad (15) $$

Since both sides of (15) are rational functions of ν, the restrictions on ν may now be relaxed. For (15) to hold, it is adequate to suppose that $2\nu-2$ is not a negative integer, n of course being a positive integer.

It is easy to derive from (12) a result which is both more general and more complicated than (15). If we apply Euler's transformation to the hypergeometric series in the expansion of $D_n^\nu(z)$ in ascending powers of $\frac{1}{2}-\frac{1}{2}z$, we find that

$$ \frac{(z^2-1)^{\nu-\frac{1}{2}}D_n^\nu(z)}{\Gamma(2\nu)} $$

$$ = \frac{\Gamma(2\nu+n)\Gamma(\frac{1}{2}-\nu)}{2\Gamma(\nu+\frac{1}{2})n!}(\tfrac{1}{2}z-\tfrac{1}{2})^{\nu-\frac{1}{2}}\,{}_2F_1(\nu+n+\tfrac{1}{2},\tfrac{1}{2}-\nu-n;\ \nu+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}z)+ $$

$$ +\frac{1}{2\nu-1}\,{}_2F_1(n+1,1-2\nu-n;\ \tfrac{3}{2}-\nu;\ \tfrac{1}{2}-\tfrac{1}{2}z). $$

We now transform (12) by using this result as well as the special case of it obtained by putting $n = 0$. In the formula thus obtained, we segregate the parts containing non-integral powers of $\frac{1}{2}-\frac{1}{2}z$ from the parts containing integral powers. This procedure yields the two formulae:

$$ \frac{\Gamma(2\nu+n)}{n!}\,{}_2F_1(\nu+n+\tfrac{1}{2},\tfrac{1}{2}-\nu-n;\nu+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}z) $$

$$ = \Gamma(2\nu)\,{}_2F_1(\nu+\tfrac{1}{2},\tfrac{1}{2}-\nu;\nu+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}z)C_n^\nu(z), \quad (16) $$

$$ \frac{1}{2\nu-1}\,{}_2F_1(n+1,1-2\nu-n;\ \tfrac{3}{2}-\nu;\ \tfrac{1}{2}-\tfrac{1}{2}z) $$

$$ = \frac{1}{2\nu-1}\,{}_2F_1(1,1-2\nu;\ \tfrac{3}{2}-\nu;\ \tfrac{1}{2}-\tfrac{1}{2}z)C_n^\nu(z)- $$

$$ -\sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]}(\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}C_{n-2m-1}^\nu(z). \quad (17) $$

Of these formulae, (16) is merely the consequence of applying Euler's transformation to (3); on the other hand, (17) is a new result.

If (17) is rewritten with the replacement of the Gegenbauer polynomials by their values in terms of hypergeometric series given by formula (3) and with z then replaced by $1-2z$, we get

$$
\begin{aligned}
{}_2F_1(n+1, 1-2\nu-n; \tfrac{3}{2}-\nu; z) &\\
= \frac{\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)}\, {}_2F_1(1, 1-2\nu; \tfrac{3}{2}-\nu; z)\, {}_2F_1(-n, 2\nu+n; \nu+\tfrac{1}{2}; z) &- \\
- \frac{1}{\Gamma(2\nu-1)} \sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]} \lambda_m \frac{\Gamma(2\nu+n-2m-1)}{(n-2m-1)!} &\times \\
\times\, {}_2F_1(1+2m-n, 2\nu+n-2m-1; \nu+\tfrac{1}{2}; z), &\qquad (18)
\end{aligned}
$$

where $$\lambda_m = (\nu+n-2m-1)\frac{(1-\nu)_m(2\nu+n-m)_m}{(n-m)_{m+1}(\nu)_{m+1}}.$$

The result of expanding both sides of (18) in ascending powers of z and equating coefficients of z^s is

$$
\begin{aligned}
&\frac{(n-1)!}{(n-s-1)!}\times \\
&\times\, {}_6F_5\begin{bmatrix} 1, \ \tfrac{1}{2}(3-\nu-n), \ 1-\nu, \ 1-2\nu-n, \ \tfrac{1}{2}(1-n+s), \ \tfrac{1}{2}(2-n+s); \\ \tfrac{1}{2}(1-\nu-n), \ 1-n, \ 1+\nu, \ \tfrac{1}{2}(3-n-s-2\nu), \ \tfrac{1}{2}(2-n-s-2\nu) \end{bmatrix} \\
&= \frac{n!\,\nu(2\nu+n+s-1)}{(n-s)!\,(2\nu-1)(\nu+n-1)}\, {}_4F_3\begin{bmatrix} 1, \ \tfrac{1}{2}-\nu-s, \ 1-2\nu, \ -s; \\ \tfrac{3}{2}-\nu, \ 1-2\nu-n-s, \ n+1-s \end{bmatrix} - \\
&\quad - \frac{(n+s)!\,\nu\Gamma(2\nu-1)}{(\nu+n-1)\Gamma(2\nu+n+s-1)}\, \frac{(2\nu+n-s)_s(\nu+\tfrac{1}{2})_s}{(\tfrac{3}{2}-\nu)_s}; \qquad (19)
\end{aligned}
$$

this result, which holds whenever ν has not a value which makes it meaningless, is the above-mentioned generalization of (15). The expression on the left of (19), which arises from Gegenbauer polynomials of the type $C^{\nu}_{n-2m-1}(z)$, must, of course, be interpreted as zero when $s \geqslant n$.

When I had discovered (19), I decided that it was worth while to try to construct a direct proof of it by means of transformations of the kinds described in W. N. Bailey's tract *Generalized Hypergeometric Series* (Cambridge, 1935). We suppose that n and s are positive integers (zero included), and, to save trouble in dealing with exceptional cases, we suppose that 2ν is not an integer; we can subsequently establish the result when 2ν has any integral value which does not make the result meaningless by an obvious limiting process.

When $s \geqslant n$, the expression on the left of (19) is, as stated above, to be interpreted as zero; the proof of (19) is then simple enough; for, when the ${}_4F_3$ on the right is written out in full as a series, the first

$s-n$ terms of the first expression on the right vanish and the remaining terms reduce to

$$\frac{n!\,\nu(2\nu+n+s-1)}{(2\nu-1)(\nu+n-1)}\,\frac{(\frac{1}{2}-\nu-s)_{s-n}(1-2\nu)_{s-n}(-s)_{s-n}}{(\frac{3}{2}-\nu)_{s-n}(1-2\nu-n-s)_{s-n}}\times$$
$$\times\,{}_3F_2\left[\begin{matrix}\frac{1}{2}-\nu-n, & 1-2\nu+s-n, & n;\\ \frac{3}{2}-\nu+s-n, & 1-2\nu-2n & \end{matrix}\right].$$

The result to be proved is therefore that

$${}_3F_2\left[\begin{matrix}\frac{1}{2}-\nu-n, & 1-2\nu+s-n, & n;\\ \frac{3}{2}-\nu+s-n, & 1-2\nu-2n & \end{matrix}\right]$$
$$=\frac{(\frac{3}{2}-\nu)_{s-n}(1-2\nu-n-s)_{s-n}}{(\frac{1}{2}-\nu-s)_{s-n}(1-2\nu)_{s-n}(-s)_{s-n}}\,\frac{(n+s)!\,\Gamma(2\nu)}{n!\,\Gamma(2\nu+n+s)}\,\frac{(2\nu+n-s)_s(\nu+\frac{1}{2})_s}{(\frac{3}{2}-\nu)_s},$$

and this is an immediate consequence of the formula of Saalschütz [Bailey, 2.2 (2)].*

With $s < n$, no such simple proof of (19) seems to exist. The obvious way to proceed is to transform the ${}_6F_5$ on the left by a formula due to Whipple [Bailey, 4.4 (5)], namely

$${}_7F_6\left[\begin{matrix}a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g;\\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g\end{matrix}\right]$$
$$=\frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-g-e)\Gamma(1+a-e-f)}\times$$
$$\times\,{}_4F_3\left[\begin{matrix}1+a-c-d, & e, & f, & g;\\ 1+a-c, & 1+a-d, & e+f+g-a\end{matrix}\right],$$

this formula being valid when the series on the right terminates and the series on the left is either a terminating series or a convergent series. In our problem we naturally take

$$g=1; \qquad e,f=\tfrac{1}{2}(1-n+s),\ \tfrac{1}{2}(2-n+s),$$

so that no questions of convergence arise; we then make

$$a\to 1-\nu-n, \qquad c\to 1-\nu, \qquad d\to 1-2\nu-n,$$

and thus, if we subsequently denote the expression on the left of (19) by W, we have now proved that W is equal to

$$\frac{(n-1)!(2\nu+n+s)(2\nu+n+s-1)}{(n-s-1)!(2\nu+2n-2)(2\nu+2s+1)}\times$$
$$\times\,{}_4F_3\left[\begin{matrix}2\nu, & \frac{1}{2}(1-n+s), & \frac{1}{2}(2-n+s), & 1;\\ 1-n, & 1+\nu, & \frac{3}{2}+s+\nu & \end{matrix}\right].$$

After reaching this point, I completed the proof of (19) in the

* Parentheses in this form are references to Bailey's tract.

special case $s = 0$ by a sequence of transformations; but the construction of the transformations necessary for the general case baffled me, and I consequently completed the proof by an entirely different method based on the theory of partial fractions. I found that partial fractions can be used to establish a large class of relations connecting functions of the type ${}_4F_3$, but the general results are somewhat nebulous, and I have decided not to publish them unless I can put them into a more concrete form. I was not satisfied with this method of proving (19), and I consequently asked Dr. Bailey whether he could deal with the problem more successfully; he soon completed the proof, starting from the last expression obtained for W, in the following manner:

In Whipple's formula [Bailey, 4.7 (1)]

$${}_5F_4\left[\begin{matrix} d, & 1-N-b-c, & -\frac{1}{2}N, & \frac{1}{2}(1-N), & 1-N-w; \\ 1-N-b, & 1-N-c, & \frac{1}{2}(1+d-w-N), & \frac{1}{2}(2+d-w-N) & \end{matrix}\right]$$
$$= \frac{(w)_N}{(w-d)_N}\,{}_4F_3\left[\begin{matrix} -N, & b, & c, & d; \\ 1-N-b, & 1-N-c, & w & \end{matrix}\right],$$

in which N is a positive integer, replace b, c, d, w, N by $s+1$, $\frac{1}{2}-\nu-n$, 1, $2-n+s-2\nu$, $n-s-1$ respectively; we thus get W equal to

$$\frac{(n-1)!(2\nu+n+s)(2\nu+n+s-1)(-2\nu)}{(n-s-1)!(2\nu+2n-2)(2\nu+2s+1)(1-n+s-2\nu)}\times$$
$$\times\,{}_4F_3\left[\begin{matrix} 1+s-n, & s+1, & \frac{1}{2}-\nu-n, & 1; \\ 1-n, & \frac{3}{2}+s+\nu, & 2-n+s-2\nu & \end{matrix}\right].$$

Next, in the formula [Bailey, 7.2 (1)]

$${}_4F_3\left[\begin{matrix} x, y, z, -N; \\ u, v, w \end{matrix}\right]$$
$$= \frac{(v-z)_N(w-z)_N}{(v)_N(w)_N}\,{}_4F_3\left[\begin{matrix} u-x, & u-y, & z, & -N; \\ 1-v+z-N, & 1-w+z-N, & u & \end{matrix}\right],$$

in which N is a positive integer and

$$u+v+w = x+y+z-N+1,$$

replace x, y, z, u, v, w, N by $s+1$, $\frac{1}{2}-\nu-n$, 1, $\frac{3}{2}+\nu+s$, $1-n$, $2-n+s-2\nu$, $n-s-1$ respectively; we thus get W equal to

$$\frac{n!(2\nu+n+s)(2\nu+n+s-1)}{(n-s-1)!(s+1)(2\nu+2n-2)(2\nu+2s+1)}\times$$
$$\times\,{}_4F_3\left[\begin{matrix} \frac{1}{2}+\nu, & 2\nu+n+s+1, & 1+s-n, & 1; \\ s+2, & 2\nu+1, & \frac{3}{2}+s+\nu & \end{matrix}\right].$$

Now this ${}_4F_3$, when written out in full, becomes

$$\sum_{m=0}^{n-s-1}\frac{(\frac{1}{2}+\nu)_m(2\nu+n+s+1)_m(1+s-n)_m}{(s+2)_m(2\nu+1)_m(\frac{3}{2}+s+\nu)_m}$$

$$=\frac{(s+1)!(2\nu-s)_{s+1}(\frac{1}{2}+\nu)_{s+1}}{(\nu-s-\frac{1}{2})_{s+1}(2\nu+n)_{s+1}(-n)_{s+1}}\times$$

$$\times\sum_{m=0}^{n-s-1}\frac{(\nu-s-\frac{1}{2})_{m+s+1}(2\nu+n)_{m+s+1}(-n)_{m+s+1}}{(m+s+1)!(2\nu-s)_{m+s+1}(\frac{1}{2}+\nu)_{m+s+1}}$$

$$=\frac{(s+1)!(2\nu-s)_{s+1}(\frac{1}{2}+\nu)_{s+1}}{(\nu-s-\frac{1}{2})_{s+1}(2\nu+n)_{s+1}(-n)_{s+1}}\sum_{m=s+1}^{n}\frac{(\nu-s-\frac{1}{2})_m(2\nu+n)_m(-n)_m}{m!(2\nu-s)_m(\frac{1}{2}+\nu)_m}.$$

Now the last $\sum$ can be written in the form

$$\sum_{m=s+1}^{n}\equiv\sum_{m=0}^{n}-\sum_{m=0}^{s},$$

and so it is equal to the difference between the sum of the complete series

$${}_3F_2\begin{bmatrix}\nu-s-\frac{1}{2}, & 2\nu+n, & -n; \\ 2\nu-s, & & \frac{1}{2}+\nu\end{bmatrix}$$

and the sum of the first $s+1$ terms of this series. Since the series is Saalschützian, its sum is

$$\frac{(\nu+\frac{1}{2})_n(-n-s)_n}{(2\nu-s)_n(\frac{1}{2}-\nu-n)_n},$$

while the sum of the first $s+1$ terms of the series (when their order is reversed) is expressible in the form

$$\frac{(\nu-s-\frac{1}{2})_s(2\nu+n)_s(-n)_s}{s!(2\nu-s)_s(\frac{1}{2}+\nu)_s}\,{}_4F_3\begin{bmatrix}1, & \frac{1}{2}-\nu-s, & 1-2\nu, & -s; \\ \frac{3}{2}-\nu, & 1-2\nu-n-s, & n+1-s\end{bmatrix}.$$

We thus find that W is equal to

$$\frac{n!(2\nu+n+s)(2\nu+n+s+1)}{(n-s-1)!(s+1)(2\nu+2n-2)(2\nu+2s+1)}\times$$

$$\times\left\{\frac{(s+1)!(2\nu-s)_{s+1}(\frac{1}{2}+\nu)_{s+1}}{(\nu-s-\frac{1}{2})_{s+1}(2\nu+n)_{s+1}(-n)_{s+1}}\,\frac{(\nu+\frac{1}{2})_n(-n-s)_n}{(2\nu-s)_n(\frac{1}{2}-\nu-n)_n}-\right.$$

$$\left.-\frac{(s+1)(2\nu)(\frac{1}{2}+\nu+s)}{(\nu-\frac{1}{2})(2\nu+n+s)(s-n)}\,{}_4F_3\begin{bmatrix}1, & \frac{1}{2}-\nu-s, & 1-2\nu, & -s; \\ \frac{3}{2}-\nu, & 1-2\nu-n-s, & n+1-s\end{bmatrix}\right\},$$

and this expression is easily reduced to the expression on the right in (19); this completes the direct proof of (19).

It may be remarked that it does not seem possible to adapt either

Dr. Bailey's method or my own method so as to yield any formula of the same type as (19) but containing more parameters.

In conclusion, I narrate the history of formula (14), so far as it concerns me. In September 1906 I became interested in Laplace's equation with n independent variables,

$$\sum_{p=1}^{n} \frac{\partial^2 V}{\partial x_p^2} = 0;$$

in addition to constructing its general solution,* on the lines of Whittaker's solutions for the cases of three and four independent variables, I noticed that a special solution is

$$V = r^m C_m^{\frac{1}{2}(n-2)}(\cos\theta),$$

where
$$r^2 = \sum_{p=1}^{n} x_p^2, \qquad r\cos\theta = x_n,$$

this solution reducing to a zonal harmonic when $n = 3$. Thereupon, on the strength of an undergraduate's knowledge of Legendre functions, I composed a lengthy account of the properties of $C_m^\nu(z)$.

I showed my work to Hobson in the following October term: he, after examining it carefully, told me that practically everything that I had done was already in print, so I put the work on one side. Formula (12), obtained by the methods of the present paper, was included in the results which I showed to Hobson, but formula (14), which I had obtained by a slightly more elaborate method than that given here, was not included because it seemed irrelevant to what were then my main objects, namely the investigation of Laplace's equation and of $C_m^\nu(z)$. I did, however, show the mere result (14) to Herman, and can well remember the characteristic phrase with which he greeted it.

Many years later (probably about 1922), when I became seriously interested in Dougall's theorem, I recollected the existence of formula (14); unfortunately, without making a proper examination of the question, I jumped to the conclusion that it was sure to be easily derivable from Dougall's theorem. This conclusion naturally then made me decide against publishing it, since I was unwilling to get involved in any questions about priority, and I was not in a position to substantiate a claim that I had discovered (14) before

* G. N. Watson, *Messenger of Math.* 36 (1907), 98–106.

8 March 1907, the date on which Dougall's paper was read to the Edinburgh Mathematical Society.

On looking into the question in July 1937, however, I realized at once that not only is (14) not a special case of Dougall's theorem, but that it is not a simple matter to prove (14) directly by means of transformations of generalized hypergeometric series. These considerations, combined with my discovery of (19), of course removed my reluctance to publish the contents of the first half of the present paper, since the lack of any close connexion between (14) and Dougall's theorem disposes of any questions concerning priority.

SOME INTEGRALS INVOLVING BESSEL FUNCTIONS

By W. N. BAILEY (*Manchester*)

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1. THE integrals

$$\int_0^{\frac{1}{2}\pi} [1-J_0(z\sin\theta)]^2 \sin^3\theta\, d\theta, \tag{1.1}$$

$$\int_0^{\frac{1}{2}\pi} [1-J_0(z\sin\theta)]^2 \frac{\{\cos(\alpha\cos\theta)-\cos\alpha\}^2}{\sin\theta}\, d\theta \tag{1.2}$$

occurred in the problem of finding the power radiated by a current sheet aerial system.* I have found that the first of these integrals can be expressed in terms of the well-known integral

$$\int_0^z J_0(t)\, dt,$$

which has been tabulated.†

I first prove this result, and then consider some further integrals which are suggested by the argument.

2. The integral (1.1) is evidently equal to

$$\tfrac{2}{3}-2I_1+I_2,$$

where

$$I_1 = \int_0^{\frac{1}{2}\pi} J_0(z\sin\theta)\sin^3\theta\, d\theta,$$

$$I_2 = \int_0^{\frac{1}{2}\pi} J_0^2(z\sin\theta)\sin^3\theta\, d\theta.$$

The value of I_1 can be obtained immediately from Sonine's first finite integral‡

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)} \int_0^{\frac{1}{2}\pi} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta\, d\theta, \tag{2.1}$$

where $R(\mu) > -1$, $R(\nu) > -1$. Taking $\mu = 0$, $\nu = \pm\frac{1}{2}$, and

* My attention was drawn to these integrals by Mr. A. Page to whom they had been communicated.

† G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), 752.

‡ Ibid., § 12.11.

subtracting the results, we find that

$$I_1 = \sqrt{\left(\frac{\pi}{2z^3}\right)}\{zJ_{\frac{1}{2}}(z)-J_{\frac{3}{2}}(z)\}$$

$$= \frac{1}{z^3}\{z\cos z-(1-z^2)\sin z\}. \tag{2.2}$$

To evaluate I_2, we use the formula*

$$J_\nu^2(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{2\nu}(2z\cos\phi)\,d\phi$$

with $\nu = 0$, and we find by (2.1) that

$$\int_0^{\frac{1}{2}\pi} J_0^2(z\sin\theta)\sin\theta\cos^{2\nu+1}\theta\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} J_0(2z\sin\theta\cos\phi)\sin\theta\cos^{2\nu+1}\theta\,d\theta d\phi$$

$$= \frac{2^{\nu+1}\Gamma(\nu+1)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{J_{\nu+1}(2z\cos\phi)}{(2z\cos\phi)^{\nu+1}}\,d\phi$$

$$= \frac{1}{2(\nu+1)}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2})_r}{(r!)^2(\nu+2)_r}z^{2r},$$

on expanding the integrand in powers of $z\cos\phi$ and integrating term by term. When $\nu = -\frac{1}{2}$ we obtain

$$\int_0^{\frac{1}{2}\pi} J_0^2(z\sin\theta)\sin\theta\,d\theta = \frac{1}{z}\int_0^z\sum_0^{\infty}\frac{(-1)^r t^{2r}\,dt}{(r!)^2} = \frac{1}{z}\int_0^z J_0(2t)\,dt. \tag{2.3}$$

When $\nu = \frac{1}{2}$ we obtain

$$\int_0^{\frac{1}{2}\pi} J_0^2(z\sin\theta)\sin\theta\cos^2\theta\,d\theta = \sum_{r=0}^{\infty}\frac{(-1)^r z^{2r}}{(r!)^2(2r+1)(2r+3)}$$

$$= \frac{1}{2}\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r}}{(r!)^2}\left\{\frac{1}{2r+1}-\frac{1}{2r+3}\right\}$$

$$= \frac{1}{2z}\int_0^z J_0(2t)\,dt - \frac{1}{2z^3}\int_0^z t^2 J_0(2t)\,dt. \tag{2.4}$$

* Watson, loc. cit., § 5.43.

From (2.3) and (2.4) we find* that

$$\int_0^{\frac{1}{2}\pi} J_0^2(z\sin\theta)\sin^3\theta\,d\theta = \frac{1}{2z}\int_0^z J_0(2t)\,dt + \frac{1}{2z^3}\int_0^z t^2 J_0(2t)\,dt$$

$$= \frac{1}{8z^2}J_0(2z) + \frac{1}{4z}J_1(2z) + \left(\frac{1}{2z} - \frac{1}{8z^3}\right)\int_0^z J_0(2t)\,dt. \tag{2.5}$$

Thus the integral I_2 is found in terms of

$$\int_0^z J_0(2t)\,dt \equiv \tfrac{1}{2}\int_0^{2z} J_0(t)\,dt.$$

A similar method shows that

$$\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = \frac{\Gamma(\frac{1}{2}-\nu)}{z\sqrt{\pi}}\int_0^z t^\nu J_\nu(2t)\,dt, \tag{2.6}$$

where $-\frac{1}{2} < \nu < \frac{1}{2}$, and

$$\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin^{2\nu+3}\theta\cos^{-2\nu}\theta\,d\theta$$

$$= \frac{\Gamma(\frac{1}{2}-\nu)}{z^3\sqrt{\pi}}\left[(\tfrac{1}{2}+\nu)z^2\int_0^z t^\nu J_\nu(2t)\,dt + (\tfrac{1}{2}-\nu)\int_0^z t^{\nu+2}J_\nu(2t)\,dt\right], \tag{2.7}$$

which reduce to (2.3) and (2.5) when $\nu = 0$.

3. The results given in § 2 suggest a consideration of the integral

$$I \equiv \int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin^{2\kappa+1}\theta\cos^{2\mu+1}\theta\,d\theta, \tag{3.1}$$

where $\nu+\kappa+1 > 0$, $\mu > -1$. Two particular cases of this integral have already been evaluated in the form of Neumann series†, the results being

$$z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin\theta\,d\theta = \sum_{n=0}^{\infty} J_{2\nu+2n+1}(2z), \tag{3.2}$$

$$z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin^{2\nu+1}\theta\,d\theta$$

$$= \frac{\Gamma(2\nu+1)}{2^{2\nu+1}\{\Gamma(\nu+1)\}^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(2\nu+n+1)}{n!\,\Gamma(2\nu+n+\frac{3}{2})}(2\nu+2n+1)J_{2\nu+2n+1}(2z). \tag{3.3}$$

* Using Watson, loc. cit., § 5.1 (4) with $\mu = 1$, $\nu = 0$.

† For (3.2) see W. N. Bailey, *Proc. London Math. Soc.* (2), 30 (1930), 415–21 (5.8), and for (3.3) see W. N. Bailey, ibid. 31 (1930), 200–8 (7.31). See also E. T. Copson, ibid. 33 (1932), 145–53.

The first of these formulae can be written in the form*

$$z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin\theta\,d\theta = \tfrac{1}{2}\int_0^{2z} J_{2\nu}(t)\,dt, \tag{3.4}$$

and this reduces to (2.3) when $\nu = 0$.

Now consider the integral (3.1). Replacing $J_\nu^2(z\sin\theta)$ by a power series† in $z\sin\theta$ and integrating term by term, we find that

$$I = \frac{1}{2}\sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(2\nu+2r+1)\Gamma(\nu+\kappa+r+1)\Gamma(\mu+1)(\frac{1}{2}z)^{2\nu+2r}}{r!\,\Gamma(2\nu+r+1)\{\Gamma(\nu+r+1)\}^2\Gamma(\nu+\kappa+\mu+r+2)}.$$

But‡

$$z^\mu = \sum_{s=0}^{\infty} \frac{(\mu+2s)\Gamma(\mu+s)}{s!} J_{\mu+2s}(2z),$$

and so

$$Iz^\alpha = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^r\Gamma(2\nu+2r+1)\Gamma(\nu+\kappa+r+1)\Gamma(\mu+1)}{2^{2\nu+2r+1}r!\,\Gamma(2\nu+r+1)\{\Gamma(\nu+r+1)\}^2}\times$$
$$\times\frac{(2\nu+2r+2s+\alpha)\Gamma(2\nu+2r+s+\alpha)}{s!\,\Gamma(\nu+\kappa+\mu+r+2)} J_{2\nu+2r+2s+\alpha}(2z).$$

Putting $s = n-r$, and summing with respect to r, we find that

$$Iz^\alpha = \frac{\Gamma(\mu+1)\Gamma(\nu+\kappa+1)}{2^{2\nu+1}\{\Gamma(\nu+1)\}^2\Gamma(\nu+\kappa+\mu+2)}\times$$
$$\times\sum_{n=0}^{\infty} \frac{(2\nu+2n+\alpha)\Gamma(2\nu+n+\alpha)}{n!} J_{2\nu+2n+\alpha}(2z)\times$$
$$\times\,{}_4F_3\left[\begin{matrix}\nu+\frac{1}{2},\ \nu+\kappa+1,\ 2\nu+n+\alpha,\ -n;\\ 2\nu+1,\ \nu+1,\ \nu+\kappa+\mu+2\end{matrix}\right].$$

The ${}_4F_3$ is Saalschützian when $\alpha = \mu+\frac{3}{2}$, and it reduces to a ${}_3F_2$ (which can be summed)|| when $\kappa = 0$ or $\kappa = \nu$. We thus obtain the formulae

$$z^{\mu+\frac{3}{2}}\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin\theta\cos^{2\mu+1}\theta\,d\theta$$
$$= \frac{\Gamma(\mu+1)}{2\Gamma(\mu+\frac{3}{2})\sqrt{\pi}}\sum_{n=0}^{\infty} \frac{\Gamma(\mu+\frac{3}{2}+n)\Gamma(\nu+\frac{1}{2}+n)\Gamma(\mu+2\nu+\frac{3}{2}+n)}{n!\,\Gamma(2\nu+1+n)\Gamma(\mu+\nu+2+n)}\times$$
$$\times(\mu+2\nu+\tfrac{3}{2}+2n)J_{\mu+2\nu+\frac{3}{2}+2n}(2z), \tag{3.5}$$

* Watson, loc. cit., § 16.56 (9). † Ibid., § 5.41. ‡ Ibid., § 5.2.

|| See W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract, 1935), § 2.2.

$$z^{\mu+\frac{3}{2}} \int\limits_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\theta)\sin^{2\nu+1}\theta\cos^{2\mu+1}\theta\,d\theta$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+\frac{1}{2})}{2\pi\Gamma(\mu+\nu+\frac{3}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)\Gamma(\mu+\nu+\frac{3}{2}+n)\Gamma(\mu+2\nu+\frac{3}{2}+n)}{n!\,\Gamma(\nu+1+n)\Gamma(\mu+2\nu+2+n)} \times$$

$$\times(\mu+2\nu+\tfrac{3}{2}+2n)J_{\mu+2\nu+\frac{3}{2}+2n}(2z). \quad (3.6)$$

When $\mu = -\frac{1}{2}$, these formulae reduce to (3.2) and (3.3).

4. In § 3 I have considered only the case when the square of a Bessel function appears in the integrand. It is natural to consider the integrals

$$\int\limits_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)J_\nu(z\sin\theta)\sin^{2\kappa+1}\theta\cos^{2\mu+1}\theta\,d\theta, \quad (4.1)$$

$$\int\limits_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)J_\nu(z\cos\theta)\sin^{2\kappa+1}\theta\cos^{2\mu+1}\theta\,d\theta. \quad (4.2)$$

The method given in § 3 can be used with only trivial modifications for (4.1). In this case the coefficients involve a series ${}_5F_4$ instead of a ${}_4F_3$, this ${}_5F_4$ being

$${}_5F_4\left[\begin{matrix}\frac{1}{2}(\lambda+\nu+1),\ \frac{1}{2}(\lambda+\nu+2),\ \frac{1}{2}(\lambda+\nu)+\kappa+1,\ \alpha+\lambda+\nu+n,\ -n;\\ \lambda+1,\ \nu+1,\ \lambda+\nu+1,\ \frac{1}{2}(\lambda+\nu)+\kappa+\mu+2\end{matrix}\right].$$

This is Saalschützian when $\alpha = \mu+\frac{3}{2}$, and can be summed whenever it reduces to a ${}_3F_2$. In this way further results can be obtained, but those given in § 3 appear to be the most interesting.

In the case of (4.2), the previous method fails to give a simple result owing to the more complicated expansion of the product of Bessel functions with different arguments. A result of a different type can, however, be obtained when $\lambda = \nu$ by using the formula*

$$J_\nu(z\cos\theta)J_\nu(z\sin\theta) = \sum_{r=0}^{\infty} \frac{(\frac{1}{4}z\sin 2\theta)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)} J_{\nu+2r}(z). \quad (4.3)$$

It then follows that

$$\int\limits_0^{\frac{1}{2}\pi} J_\nu(z\cos\theta)J_\nu(z\sin\theta)\sin^{2\kappa+1}\theta\cos^{2\mu+1}\theta\,d\theta$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}\nu+\kappa+1+r)\Gamma(\frac{1}{2}\nu+\mu+1+r)}{2\,r!\,\Gamma(\nu+r+1)\Gamma(\nu+\kappa+\mu+2+2r)} (\tfrac{1}{2}z)^{\nu+2r} J_{\nu+2r}(z). \quad (4.4)$$

* W. N. Bailey, *Proc. London Math. Soc.* (2), 41 (1936), 215–20 (5.2).

It will be noticed that this formula contains one more parameter than (3.5) and (3.6). A result more in keeping with those of § 3 is obtained by using the formula*

$$(\tfrac{1}{2}z)^{\mu-\nu}J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\mu+2r)\Gamma(\mu+r)(\mu-\nu)_r}{r!\,\Gamma(\nu+r+1)} J_{\mu+2r}(z)$$

to transform (4.4), but the coefficients obtained are even more complicated than in the previous cases, and any formula in which the coefficients simplify would possess very little generality.

[*Added* 9 *March* 1938.] An integral analogous to (4.2), but involving Bessel functions with different z, can be obtained from Sonine's second finite integral†

$$\int_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)J_\nu(Z\cos\theta)\sin^{\lambda+1}\theta\cos^{\nu+1}\theta\,d\theta = \frac{z^\lambda Z^\nu J_{\lambda+\nu+1}\{\sqrt{(Z^2+z^2)}\}}{(Z^2+z^2)^{\frac{1}{2}(\lambda+\nu+1)}}. \tag{4.5}$$

We multiply (4.5) by $z^\lambda Z^\nu$ and operate on the result with

$$\left(\frac{d}{zdz}\right)^m\left(\frac{d}{ZdZ}\right)^n,$$

using the formulae

$$\left(\frac{d}{zdz}\right)^m\{z^\lambda J_\lambda(z\sin\theta)\} = \sin^m\theta\, z^{\lambda-m}J_{\lambda-m}(z\sin\theta),$$

$$\left(\frac{d}{ZdZ}\right)^n\{Z^\nu J_\nu(Z\cos\theta)\} = \cos^n\theta\, Z^{\nu-n}J_{\nu-n}(Z\cos\theta).$$

Then, changing λ, ν into $\lambda+m$, $\nu+n$, we obtain the formula

$$z^\lambda Z^\nu \int_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)J_\nu(Z\cos\theta)\sin^{\lambda+2m+1}\theta\cos^{\nu+2n+1}\theta\,d\theta$$

$$= \left(\frac{d}{zdz}\right)^m\left(\frac{d}{ZdZ}\right)^n\left[\frac{z^{2\lambda+2m}Z^{2\nu+2n}J_{\lambda+\nu+m+n+1}\{\sqrt{(Z^2+z^2)}\}}{(Z^2+z^2)^{\frac{1}{2}(\lambda+\nu+m+n+1)}}\right]. \tag{4.6}$$

This shows, in particular, that the integral (4.2) can be evaluated in finite terms when $\kappa-\frac{1}{2}\lambda$, $\mu-\frac{1}{2}\nu$ are positive integers or zero.

* Watson, loc. cit., § 5.21. † Ibid. § 12.13.

Again, using the expansion* of $J_\nu(z)$ as a series of the form $\sum a_n z^{\nu-\mu+m} J_{\mu+m}(z)$, we find that

$$J_\lambda(z\sin\theta)J_\nu(Z\cos\theta)(z\sin\theta)^{2\kappa+1}(Z\cos\theta)^{2\mu+1}$$
$$= 2^{2\kappa+2\mu+2}\Gamma(\kappa+1+\tfrac{1}{2}\lambda)\Gamma(\mu+1+\tfrac{1}{2}\nu)\times$$
$$\times\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\tfrac{1}{2}\lambda-\kappa)_r(\tfrac{1}{2}\nu-\mu)_s}{r!\,s!\,\Gamma(\lambda+r+1)\Gamma(\nu+s+1)}J_{\kappa+\frac{1}{2}\lambda+r}(z\sin\theta)\times$$
$$\times J_{\mu+\frac{1}{2}\nu+s}(Z\cos\theta)(\tfrac{1}{2}z\sin\theta)^{\kappa+1+\frac{1}{2}\lambda+r}(\tfrac{1}{2}Z\cos\theta)^{\mu+1+\frac{1}{2}\nu+s},$$

and, integrating with respect to θ, and using (4.5), we obtain

$$\int_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)J_\nu(Z\cos\theta)\sin^{2\kappa+1}\theta\cos^{2\mu+1}\theta\,d\theta$$
$$= 2^{\kappa+\mu-\frac{1}{2}\lambda-\frac{1}{2}\nu}\Gamma(\kappa+1+\tfrac{1}{2}\lambda)\Gamma(\mu+1+\tfrac{1}{2}\nu)\times$$
$$\times\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\tfrac{1}{2}\lambda-\kappa)_r(\tfrac{1}{2}\nu-\mu)_s}{2^{r+s}r!\,s!\,\Gamma(\lambda+r+1)\Gamma(\nu+s+1)}\times$$
$$\times\frac{z^{\lambda+2r}Z^{\nu+2s}J_{\kappa+\mu+1+\frac{1}{2}\lambda+\frac{1}{2}\nu+r+s}\{\sqrt{(Z^2+z^2)}\}}{(Z^2+z^2)^{\frac{1}{2}(\kappa+\mu+1+\frac{1}{2}\lambda+\frac{1}{2}\nu+r+s)}}. \tag{4.7}$$

It is again evident that, when $\kappa-\frac{1}{2}\lambda$, $\mu-\frac{1}{2}\nu$ are positive integers or zero, this double series terminates and, in particular, the integral (4.2) is given in finite terms.

* Ibid. § 5.23.

A THEORY OF GENERAL TRANSFORMS FOR FUNCTIONS OF THE CLASS $L^p(0, \infty)$ $(1 < p \leqslant 2)$

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1. Introduction. The object of this paper is to show that Titchmarsh's theory of Fourier transforms for functions of the class $L^p(0,\infty)$† is a particular case of a theory which is true for certain of Watson's 'general transforms'.‡ The theorems will be stated for generalized Hankel transforms, those for generalized Fourier transforms being obtained by putting $\lambda = \frac{1}{2}$.

We shall require the following theorem. The case $\lambda = \frac{1}{2}$ ($\chi_\lambda(x)$ complex) has been proved by Kober,§ and the proof of the general theorem presents few additional difficulties.

THEOREM 1. *Let $\chi_\lambda(x)$ satisfy the following conditions:*

1 *a.* *$\chi_\lambda(x)/x$ belongs to $L^2(0,\infty)$ and satisfies the equation*

$$\int_0^\infty \frac{\chi_\lambda(xy)\chi_\lambda(yz)}{y^2}\,dy = \begin{cases} \pm\dfrac{1}{2\lambda}(xz)^{\frac{1}{2}}\left(\dfrac{z}{x}\right)^{\pm\lambda} & (x > z), \\[2ex] \pm\dfrac{1}{2\lambda}(xz)^{\frac{1}{2}}\left(\dfrac{x}{z}\right)^{\pm\lambda} & (x < z), \end{cases} \tag{1.1}$$

for every x and z in $(0,\infty)$, the upper or lower signs being taken according as $\Re(\lambda) > 0$ or < 0.

1 *b.* *$\chi_\lambda(x)$ is an integral, so that a function $\omega_\lambda(x)$ exists almost everywhere such that*||

$$\chi_\lambda(x) = \lim_{\delta\to 0} x^{-\lambda+\frac{1}{2}} \int_\delta^x t^{\lambda-\frac{1}{2}}\omega_\lambda(t)\,dt \quad (\Re(\lambda) > 0),$$

$$\chi_\lambda(x) = -\lim_{X\to\infty} x^{-\lambda+\frac{1}{2}} \int_x^X t^{\lambda-\frac{1}{2}}\omega_\lambda(t)\,dt \quad (\Re(\lambda) < 0).$$

Then, if $f(x)$ belong to $L^2(0,\infty)$, the function

$$g(x) = \underset{\substack{a\to 0\\ b\to\infty}}{\mathrm{l.i.m.}} \int_a^b f(y)\omega_\lambda(xy)\,dy = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(xy)}{y}f(y)\,dy\right\} \tag{1.2}$$

† *Proc. London Math. Soc.* (2) 23 (1924), 279–89.

‡ Ibid. 35 (1933), 156–99.

§ *Quart. J. of Math.* (Oxford) 8 (1937), 172–85. Since the publication of Kober's two papers in vol. 8 of the *Quarterly Journal*, I have completely revised and extended this paper.

|| Since $\chi_\lambda(x)/x$ belongs to $L^2(0,\infty)$, it follows that $x^{\lambda-\frac{1}{2}}\chi_\lambda(x)$ tends to zero as $x \to 0$ $(\Re(\lambda) > 0)$ and as $x \to \infty$ $(\Re(\lambda) < 0)$.

exists almost everywhere and belongs to $L^2(0,\infty)$. *The relation between* $f(x)$ *and* $g(x)$ *is a reciprocal one, so that*

$$f(x) = \underset{\substack{a\to 0\\ b\to\infty}}{\text{l.i.m.}} \int_a^b g(y)\omega_\lambda(xy)\,dy = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(xy)}{y}g(y)\,dy\right\}. \quad (1.3)$$

Further, a constant $A_\lambda\ (\geqslant 1)$ *exists*† *such that*

$$\int_0^\infty |g(x)|^2\,dx \leqslant A_\lambda \int_0^\infty |f(x)|^2\,dx, \qquad \int_0^\infty |f(x)|^2\,dx \leqslant A_\lambda \int_0^\infty |g(x)|^2\,dx. \quad (1.4)$$

Finally, if $f_1(x)$ *and* $f_2(x)$ *both belong to* $L^2(0,\infty)$ *and if* $g_1(x)$, $g_2(x)$ *are their transforms, then*

$$\int_0^\infty f_1(x)g_2(x)\,dx = \int_0^\infty f_2(x)g_1(x)\,dx.$$

In this theorem the equations given in $1a$ and $1b$ are plainly exact, but (1.2), (1.3), and (1.4) are only true 'almost everywhere'. As it will be obvious which equations are exact, we shall use the sign '$=$' throughout. Unless it is otherwise stated, p will represent a real number such that $1 < p \leqslant 2$ and p' will be its conjugate, so that $1/p+1/p' = 1$. We shall use $\underset{T\to\infty}{\mathfrak{L}_r}$ for 'the limit in mean with exponent r as $T\to\infty$'. When $f(x)$ and $g(x)$ are related by the equations (1.2) and (1.3), we shall say that they are χ_λ-transforms of one another. All the functions with which we are concerned may be complex functions of a real variable.

The following is our main theorem:

THEOREM 2. *Let* $\Re(\lambda) > 0$ *or* $< -\frac{1}{2}$ *and let* $\chi_\lambda(x)$ *satisfy condition* $1\,a$ *and the following conditions*:

$2\,a$. $\chi_\lambda(x)$ *is an integral, so that a function* $\omega_\lambda(x)$ *exists almost everywhere such that*

$$\chi_\lambda(x) = x^{-\lambda+\frac{1}{2}}\int_0^x t^{\lambda-\frac{1}{2}}\omega_\lambda(t)\,dt \quad (\Re(\lambda) > 0),$$

$$\chi_\lambda(x) = -x^{-\lambda+\frac{1}{2}}\int_x^\infty t^{\lambda-\frac{1}{2}}\omega_\lambda(t)\,dt \quad (\Re(\lambda) < -\tfrac{1}{2});$$

† $A_\lambda \geqslant |\Omega_\lambda(\frac{1}{2}+it)|$, where

$$\frac{\Omega_\lambda(\frac{1}{2}+it)}{\lambda-it} = \underset{a\to\infty}{\text{l.i.m.}} \int_{a^{-1}}^a \chi_\lambda(x)x^{-\frac{3}{2}+it}\,dx.$$

When λ and $\chi_\lambda(x)$ are real, $A_\lambda = |\Omega_\lambda(\frac{1}{2}+it)| = 1$.

2*b*. *a number* m_λ $(<\infty)$ *exists such that* $|\omega_\lambda(x)| \leqslant m_\lambda$ *whenever* $\omega_\lambda(x)$ *exists;*

2*c*. $\chi_\lambda(x)/x$ *belongs to* $L^{p_1}(0,\infty)$, *where* $1 < p_1 < 2$.
Then, if $f(x)$ *belong to* $L^p(0,\infty)$, *the function*

$$g(x) = \underset{a\to\infty}{\mathfrak{L}_{p'}} \int\limits_0^a f(y)\omega_\lambda(xy)\,dy = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int\limits_0^\infty \frac{\chi_\lambda(xy)}{y} f(y)\,dy\right\} \tag{1.5}$$

exists almost everywhere and belongs to $L^{p'}(0,\infty)$ *and*

$$\left(\int\limits_0^\infty |g(x)|^{p'}\,dx\right)^{1/p'} \leqslant A_\lambda^{1/p'} m_\lambda^{(2-p)/p}\left(\int\limits_0^\infty |f(x)|^p\,dx\right)^{1/p}. \tag{1.6}$$

If $f_1(x)$ *and* $f_2(x)$ *both belong to* $L^p(0,\infty)$ *and* $g_1(x)$, $g_2(x)$ *are their* χ_λ*-transforms, then*

$$\int\limits_0^\infty f_1(x)g_2(x)\,dx = \int\limits_0^\infty f_2(x)g_1(x)\,dx. \tag{1.7}$$

If $p \geqslant p_1$, *we also have*

$$f(x) = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int\limits_0^\infty \frac{\chi_\lambda(xy)}{y} g(y)\,dy\right\}, \tag{1.8}$$

which is the equation reciprocal to (1.5).

Titchmarsh's theorem for cosine transforms is obtained from this by taking $\lambda = \frac{1}{2}$,

$$\omega_\lambda(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos x, \qquad \chi_\lambda(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x, \qquad m_\lambda = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

For this kernel $A_\lambda = 1$ and the conditions of the theorem are plainly satisfied for any $p_1 > 1$. In this case, Hille and Tamarkin† have shown that (1.8) may be replaced by the equation

$$f(x) = \underset{a\to\infty}{\mathfrak{L}_p} \int\limits_0^a g(y)\omega_\lambda(xy)\,dy.$$

I have not been able to prove it true in the general case without making very drastic assumptions about

$$\int\limits_0^a \omega_\lambda(xt)\omega_\lambda(yt)\,dt.$$

† *Bull. American Math. Soc.* 39 (1933), 768–74.

Two other kernels, for which the theorem is true, are given by

$$\lambda = \tfrac{1}{2},\quad \omega_\lambda(x) = J_0(2x^{\frac{1}{2}}),\quad \chi_\lambda(x) = x^{\frac{1}{2}}J_1(2x^{\frac{1}{2}}),\quad m_\lambda \leqslant 1,\quad p_1 > \tfrac{4}{3};$$

$$\lambda = \tfrac{1}{4},\quad \omega_\lambda(x) = \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\sin(2x^{\frac{1}{2}}),\quad \chi_\lambda(x) = \pi^{-\frac{1}{2}}x^{\frac{1}{4}}\{1-\cos(2x^{\frac{1}{2}})\},$$
$$m_\lambda < 1,\quad p_1 > \tfrac{4}{3}.$$

In each case $A_\lambda = 1$. These will be needed in § 4, where we shall consider kernels which do not satisfy the conditions of Theorem 2.

2. Preliminary lemmas. We shall prove Theorem 2 with the help of some convexity theorems of M. Riesz;† these are stated in Lemmas 1 and 2.

Definition. Let $T = T(f)$ be a transformation of $f(x)$ such that, for any constants μ_1 and μ_2,

$$T(\mu_1 f_1+\mu_2 f_2) = \mu_1 T(f_1)+\mu_2 T(f_2),$$

and such that the transforms of functions of the class $L^q(a,b)$ make part or the whole of a certain class $L^r(c,d)$. Suppose also that a number $M^*_{\alpha,\gamma}$ (where $\alpha = q^{-1}$, $\gamma = r^{-1}$) exists such that always

$$\left(\int_c^d |T(f)|^r\,dx\right)^{1/r} \leqslant M^*_{\alpha,\gamma}\left(\int_a^b |f|^q\,dx\right)^{1/q}, \tag{2.1}$$

$M^*_{\alpha,\gamma}$ denoting the least of all such possible numbers. Then we say that T is a linear functional transformation of $L^q(a,b)$ into $L^r(c,d)$.

LEMMA 1. *Let T be a linear functional transformation of certain classes $L^q(a,b)$ into certain classes $L^r(c,d)$, the relation between q and r being such that $r \geqslant q$ and that the point (α,γ), where $\alpha = q^{-1}$, $\gamma = r^{-1}$, describes a segment in the plane (α,γ). Then $\log M^*_{\alpha,\gamma}$ is a convex function of the points of the segment.*

LEMMA 2. *Whenever there is a linear functional transformation from $L^{q_1}(a,b)$ to $L^{r_1}(c,d)$ and from $L^{q_2}(a,b)$ to $L^{r_2}(c,d)$, with $q_1 \leqslant r_1$ and $q_2 \leqslant r_2$, the transformation can be extended to all pairs of exponents corresponding to the points (α,γ) of the segment joining the points (α_1,γ_1) and (α_2,γ_2).*

From these lemmas we can deduce

LEMMA 3. *Let $T(f)$ be a linear functional transformation of $f(x)$ such that*

$$\int_c^d |T(f)|^2\,dx \leqslant A\int_a^b |f|^2\,dx \tag{2.2}$$

† *Acta Math.* 49 (1926–7), 465–97.

for all $f(x)$ of $L^2(a,b)$, and such that

$$\max_{c \leqslant x \leqslant d} |T(f)| \leqslant m \int_a^b |f|\, dx \tag{2.3}$$

for all $f(x)$ of $L(a,b)$. Then

$$\left(\int_c^d |T(f)|^{p'} dx \right)^{1/p'} \leqslant A^{1/p'} m^{(2-p)/p} \left(\int_a^b |f|^p\, dx \right)^{1/p} \tag{2.4}$$

for all functions $f(x)$ of $L^p(a,b)$.

To prove this, consider the segment joining the points $(\frac{1}{2},\frac{1}{2})$, $(1,0)$ in the plane (α,γ). Any point of this segment has the coordinates $(\alpha, 1-\alpha)$ where $\frac{1}{2} \leqslant \alpha \leqslant 1$. It therefore follows from Lemmas 1 and 2 that

$$M^*_{\alpha,1-\alpha} \leqslant (M^*_{\frac{1}{2},\frac{1}{2}})^{2(1-\alpha)} (M^*_{1,0})^{2\alpha-1}.$$

But from (2.2) and (2.3) we have

$$M^*_{\frac{1}{2},\frac{1}{2}} \leqslant A^{\frac{1}{2}}, \qquad M^*_{1,0} \leqslant m;$$

hence, if $\alpha = p^{-1}$ (so that $1 \leqslant p \leqslant 2$),

$$M^*_{\alpha,1-\alpha} \leqslant A^{1/p'} m^{(2-p)/p},$$

and this with (2.1) implies (2.4).

Finally we shall need

LEMMA 4. *If $\chi_\lambda(x)$ satisfy conditions 1 a, 2 a, and 2 b, then $\chi_\lambda(x)/x$ belongs to $L^r(0,\infty)$ for every $r \geqslant 2$.*

For it follows from 2 a and 2 b that

$$\left| \frac{\chi_\lambda(x)}{x} \right| \leqslant \frac{m_\lambda}{|\Re(\lambda)+\frac{1}{2}|}.$$

Hence, if $r \geqslant 2$,

$$\int_0^\infty \left| \frac{\chi_\lambda(x)}{x} \right|^r dx \leqslant \frac{m_\lambda^{r-2}}{|\Re(\lambda)+\frac{1}{2}|^{r-2}} \int_0^\infty \left| \frac{\chi_\lambda(x)}{x} \right|^2 dx < A \quad \text{(say)},$$

since $\chi_\lambda(x)/x$ belongs to $L^2(0,\infty)$. This proves the lemma.

3. The proof of Theorem 2. Let $F(x)$ be any function of $L^2(0,\infty)$ and let

$$F_a(x) = \begin{cases} F(x) & (0 \leqslant x \leqslant a), \\ 0 & (x > a), \end{cases}$$

$F_b(x)$ having a similar meaning. Let $G_a(x)$ be the χ_λ-transform of $F_a(x)$ so that

$$G_a(x) = \int_0^a F(y)\omega_\lambda(xy)\, dy.$$

This follows from Theorem 1, the limit in mean being omitted since the right-hand side exists as an L-integral in virtue of Schwarz's inequality and (2 b). Thus

$$G_b(x)-G_a(x) = \int_a^b F(y)\omega_\lambda(xy)\,dy, \tag{3.1}$$

and by (1.4) we have

$$\int_0^\infty |G_b(x)-G_a(x)|^2\,dx \leqslant A_\lambda \int_a^b |F(x)|^2\,dx. \tag{3.2}$$

If, however, $F(x)$ belongs to $L(0,\infty)$, it follows from (2 b) and (3.1), which will again exist, that

$$\max_{0\leqslant x\leqslant\infty} |G_b(x)-G_a(x)| \leqslant m_\lambda \int_a^b |F(y)|\,dy. \tag{3.3}$$

From (3.2) and (3.3) it follows that (3.1) defines a linear functional transformation which satisfies the conditions of Lemma 3. If therefore $f(x)$ is any function of $L^p(0,\infty)$, we shall have

$$\left(\int_0^\infty |g_b(x)-g_a(x)|^{p'}\,dx\right)^{1/p'} \leqslant A_\lambda^{1/p'} m_\lambda^{(2-p)/p}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \tag{3.4}$$

where $$g_a(x) = \int_0^a f(y)\omega_\lambda(xy)\,dy.$$

The right-hand side of (3.4) may be made less than ϵ by taking $b > a \geqslant a_0(\epsilon)$ and it follows that $g_a(x)$ converges in mean with exponent p', as $a\to\infty$, to a function $g(x)$ of $L^{p'}(0,\infty)$. This proves the first part of (1.5).

Now, when $\Re(\lambda) > 0$, $x^{\lambda-\frac{1}{2}}$ belongs to $L^p(0,y)$ and we therefore have, by the properties of limits in mean,

$$\begin{aligned}\int_0^y x^{\lambda-\frac{1}{2}}g(x)\,dx &= \lim_{a\to\infty}\int_0^y x^{\lambda-\frac{1}{2}}\,dx\int_0^a f(t)\omega_\lambda(xt)\,dt \\ &= \lim_{a\to\infty}\int_0^a f(t)\,dt\int_0^y x^{\lambda-\frac{1}{2}}\omega_\lambda(xt)\,dx \\ &= y^{\lambda-\frac{1}{2}}\int_0^\infty \frac{\chi_\lambda(yt)}{t}f(t)\,dt \end{aligned}\tag{3.5}$$

by (2 a). This exists, since (by Lemma 4) $\chi_\lambda(t)/t$ belongs to $L^{p'}(0,\infty)$.

When $\Re(\lambda) < -\frac{1}{2}$, $x^{\lambda-\frac{1}{2}}$ belongs to $L^p(y,\infty)$ and we have similarly

$$-\int_y^\infty x^{\lambda-\frac{1}{2}} g(x)\,dx = y^{\lambda-\frac{1}{2}} \int_0^\infty \frac{\chi_\lambda(yt)}{t} f(t)\,dt. \tag{3.6}$$

The second part of (1.5) now follows from (3.5) and (3.6).

The inequality (1.6) follows at once from (3.4) by taking $a = 0$ and letting $b \to \infty$. Equation (1.7) is proved in the usual way; we have (with an obvious notation)

$$\begin{aligned}\int_0^b f_1(x) g_{2,a}(x)\,dx &= \int_0^b f_1(x)\,dx \int_0^a f_2(y)\omega_\lambda(xy)\,dy \\ &= \int_0^a f_2(y)\,dy \int_0^b f_1(x)\omega_\lambda(xy)\,dx \\ &= \int_0^a f_2(y) g_{1,b}(y)\,dy.\end{aligned}$$

Equation (1.7) now follows from this, by the properties of limits in mean, on making first a and then b tend to infinity.

So far no use has been made of condition 2 c and our analysis has been true for $1 < p \leqslant 2$. The extra condition is required in the proof of (1.8). By 2 c and Lemma 4 we see that $\chi_\lambda(x)/x$ belongs to $L^r(0,\infty)$ for $r \geqslant p_1$ and therefore to $L^p(0,\infty)$ $(p \geqslant p_1)$. Consider the equation (1.1); when $\Re(\lambda) > 0$, this may be written

$$x^{-\lambda+\frac{1}{2}} \frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}} \int_0^\infty \frac{\chi_\lambda(xy)}{y} \frac{\chi_\lambda(yz)}{y}\,dy\right\} = \begin{cases} 0 & (x > z), \\ (x/z)^{\lambda-\frac{1}{2}} & (x < z). \end{cases} \tag{3.7}$$

Hence the χ_λ-transform of $\chi_\lambda(yz)/y$, which belongs to $L^p(0,\infty)$ for $p \geqslant p_1$, is the function

$$\phi(x) = \begin{cases} 0 & (x > z), \\ (x/z)^{\lambda-\frac{1}{2}} & (x < z), \end{cases}$$

and therefore, if $f(x)$ belongs to $L^p(0,\infty)$ $(p \geqslant p_1)$, we have by (1.7)

$$\begin{aligned}\int_0^\infty \frac{\chi_\lambda(yz)}{y} g(y)\,dy &= \int_0^\infty \phi(x) f(x)\,dx \\ &= \int_0^z (x/z)^{\lambda-\frac{1}{2}} f(x)\,dx,\end{aligned}$$

and thus
$$f(z) = z^{-\lambda+\frac{1}{2}} \frac{d}{dz}\left\{z^{\lambda-\frac{1}{2}} \int_0^\infty \frac{\chi_\lambda(yz)}{y} g(y)\,dy\right\}.$$

When $\Re(\lambda) < -\frac{1}{2}$, (3.7) is changed to

$$x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int\limits_0^\infty \frac{\chi_\lambda(xy)}{y}\frac{\chi_\lambda(yz)}{y}\,dy\right\} = \begin{cases}-(x/z)^{\lambda-\frac{1}{2}} & (x > z),\\ 0 & (x < z),\end{cases}$$

and thus by (1.7),

$$\int\limits_0^\infty \frac{\chi_\lambda(yz)}{y}g(y)\,dy = -\int\limits_z^\infty \left(\frac{x}{z}\right)^{\lambda-\frac{1}{2}} f(x)\,dx,$$

so that (1.8) is again true.

4. Theorem 2 is the simplest theorem of its kind and it covers a large number of cases. The following theorem is more general, but it is difficult to apply; from it we shall deduce a useful theorem which covers many cases not covered by Theorem 2.

THEOREM 3. *Let $\Re(\lambda) > 0$ or $< \frac{1}{2}-1/p_1$ $(p_1 > 1)$ and let $\chi_\lambda(x)$ satisfy condition* 1 *a, and also the following conditions:*

3 *a. the function*

$$g(x) = x^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{x^{\lambda-\frac{1}{2}}\int\limits_0^\infty \frac{\chi_\lambda(xy)}{y}f(y)\,dy\right\} \tag{4.1}$$

exists for every $f(x)$ of $L^{p_1}(0,\infty)$ and satisfies the inequality

$$\left(\int\limits_0^\infty |g(x)|^{p_1'}\,dx\right)^{1/p_1'} \leqslant C_{p_1,\lambda}\left(\int\limits_0^\infty |f(x)|^{p_1}\,dx\right)^{1/p_1}; \tag{4.2}$$

3 *b. $\chi_\lambda(x)/x$ belongs to $L^{p_1}(0,\infty)$.*

Then, if $f(x)$ belong to $L^p(0,\infty)$, where $p_1 \leqslant p \leqslant 2$, the function defined by (4.1) *exists almost everywhere and belongs to $L^{p'}(0,\infty)$ and satisfies the inequality*†

$$\left(\int\limits_0^\infty |g(x)|^{p'}\,dx\right)^{1/p'} \leqslant C_{p,\lambda}\left(\int\limits_0^\infty |f(x)|^{p}\,dx\right)^{1/p}. \tag{4.3}$$

Equations (1.7) *and* (1.8) *are also true.*

Equation (4.2) implies that $\chi_\lambda(x)/x$ belongs to $L^{p_1'}(0,\infty)$, for, if $\Re(\lambda) > 0$, take

$$f(x) = \begin{cases} x^{\lambda-\frac{1}{2}} & (0 \leqslant x \leqslant 1),\\ 0 & (x > 1),\end{cases} \tag{4.4}$$

in (4.1); then $g(x) = \chi_\lambda(x)/x$. If $\Re(\lambda) < \frac{1}{2}-1/p_1$, take

$$f(x) = \begin{cases} 0 & (0 \leqslant x < 1),\\ x^{\lambda-\frac{1}{2}} & (x \geqslant 1),\end{cases} \tag{4.5}$$

† Actually $C_{p,\lambda} = A_\lambda^{(p-p_1)/p(2-p_1)}C_{p_1,\lambda}^{p_1(2-p)/p(2-p_1)}$.

and again $g(x) = \chi_\lambda(x)/x$. In each case $f(x)$ belongs to $L^{p_1}(0,\infty)$, and therefore, by (4.2), $\chi_\lambda(x)/x$ belongs to $L^{p_1'}(0,\infty)$.

The proof of Theorem 3 is similar to that of Theorem 2 (with obvious modifications), and so we shall omit it.

The restrictions on λ in Theorems 2 and 3 arise quite naturally. In order that $\chi_\lambda(x)$ may be a kernel for which the L^2 transform theory holds, we must have $\Re(\lambda) > 0$ or < 0; then the functions defined by (4.4) and (4.5) both belong to $L^2(0,\infty)$. When we are considering a theory for functions of L^{p_1}, these functions must belong to $L^{p_1}(0,\infty)$ and we therefore need $\Re(\lambda) > \frac{1}{2}-1/p_1$ or $< \frac{1}{2}-1/p_1$. Since, however, both L^2 and L^{p_1} theories must exist simultaneously, we must have $\Re(\lambda) > 0$ or $< \frac{1}{2}-1/p_1$. Theorem 2 is the case $p_1 = 1$.

In a recent paper Kober† has worked out the complete L^p theory for Hankel transforms, and it is interesting to see how his restrictions on ν and p arise. In this case

$$\lambda = \nu+1, \qquad \chi_\lambda(x) = x^{\frac{1}{2}}J_{\nu+1,l}(x), \qquad \omega_\lambda(x) = x^{\frac{1}{2}}J_{\nu,l}(x),$$

where $J_{\nu,l}(x)$ is the 'cut' Bessel function defined by

$$J_{\nu,l}(x) = \sum_{k=l}^{\infty} \frac{(-1)^k(\frac{1}{2}x)^{\nu+2k}}{\Gamma(k+1)\Gamma(\nu+k+1)},$$

and $\quad l = [\frac{1}{2}(1-\Re(\nu))] \;\; (\Re(\nu) < -1), \quad l = 0 \;\; (\Re(\nu) > -1).$

There is an L^2 theory if $\Re(\lambda) = \Re(\nu)+1 > 0$, or if $\Re(\lambda) < 0$ and $l = l(2;\nu)$ is determined by the condition

$$0 < \Re(\lambda)+2l < 2. \tag{4.6}$$

When considering an L^p theory, Kober first proves an inequality similar to (4.2) and his proof of this is valid if

(i) $\Re(\lambda) > p_1^{-1}-\frac{1}{2} > 0$, or

(ii) $\Re(\lambda) < \frac{1}{2}-p_1^{-1}$ and congruent, to modulus 2, to a number lying between $p_1^{-1}-\frac{1}{2}$ and $\frac{3}{2}+p_1^{-1}$.

In case (i) $l = 0$, and in case (ii) $l = l(p_1,\nu)$ is uniquely determined by the condition

$$p_1^{-1}-\tfrac{1}{2} < \Re(\lambda)+2l < \tfrac{3}{2}+p_1^{-1}.$$

Condition $3b$, however, is only satisfied by $\chi_\lambda(x) = x^{\frac{1}{2}}J_{\nu+1,l}(x)$ if $\Re(\lambda) > p_1^{-1}-\frac{1}{2}$ or if

$$p_1^{-1}-\tfrac{1}{2} < \Re(\lambda)+2l < \tfrac{5}{2}-p_1^{-1}, \tag{4.7}$$

† *Quart. J. of Math.* (Oxford), 8 (1937), 186–99.

so the possible values of λ, when $\mathfrak{R}(\lambda) < 0$, have to be further restricted. Since

$$0 < p_1^{-1}-\tfrac{1}{2} < \tfrac{5}{2}-p_1^{-1} < 2,$$

it follows from (4.6) and (4.7) that $l(2;\nu) = l(p_1;\nu)$ when λ satisfies the condition (4.7), and thus (by Theorem 3) that the transform theory will hold for $p_1 \leqslant p \leqslant 2$. Replacing $\mathfrak{R}(\lambda)$ by $\mathfrak{R}(\nu)+1$, the conditions for ν and l become

(i)′ $\mathfrak{R}(\nu) > p_1^{-1}-\frac{3}{2}$, $\quad l = 0$;

(ii)′ $p_1^{-1}-\frac{3}{2} < \mathfrak{R}(\nu)+2l < \frac{3}{2}-p_1^{-1}$, $\quad l = [\frac{1}{2}(1-\mathfrak{R}(\nu))] > 0$;

and these are the conditions assumed by Kober in his paper.

With the assistance of a general theorem, which is proved by Kober in the same paper, we are able to obtain fairly simple conditions for the truth of condition $3a$. The form in which we shall use the theorem is given in the following lemma.

Lemma 5. *Let $L(x)$ be an integrable function for $0 < x < \infty$ and let it satisfy the following conditions:*

$$|L(x)| < Ax^{-\eta} \quad (x \leqslant 1),$$
$$|L(x)| < Ax^{-\rho} \quad (x \geqslant 1)$$

where $\eta < 1-p^{-1} < \rho$. Then, if $f(x)$ belong to $L^p(0,\infty)$ and if

$$\phi_a(x) = \int_0^a f(t)L(xt)\,dt,$$

a constant K_p exists, which depends only on $L(x)$, such that

$$\left(\int_0^\infty |\phi_b(x)-\phi_a(x)|^{p'}\,dx\right)^{1/p'} \leqslant K_p\left(\int_a^b |f(x)|^p\,dx\right)^{1/p}.$$

From this lemma and Theorems 2 and 3 we can deduce

Theorem 4. *Let $\mathfrak{R}(\lambda) > 0$ or $< \frac{1}{2}-p_1^{-1}$ and let $\chi_\lambda(x)$ satisfy conditions $1a$, $1b$, and the following conditions:*

$4a$. $$\omega_\lambda(x) = \sum_{k=1}^n a_k\,\omega_{\lambda_k}(x)+L(x),$$

where $\omega_{\lambda_k}(x)$ $(k = 1,\ldots,n)$ are derived from functions $\chi_{\lambda_k}(x)$ satisfying conditions $1a$, $1b$, and $2b$, $a_1,\ldots, a_n$ are constants, and $L(x)$ satisfies the conditions of Lemma 5 when $p = p_1$;

$4b$. *$\chi_\lambda(x)/x$ belongs to $L^r(0,\infty)$ for $p_1 \leqslant r \leqslant p_1'$.*

Then, if $f(x)$ belong to $L^p(0,\infty)$ $(p_1 \leqslant p \leqslant 2)$, the function $g(x)$, defined by (1.5), exists almost everywhere, belongs to $L^{p'}(0,\infty)$, and satisfies the inequality (4.3). Equations (1.7) and (1.8) are also true.

In order to prove this theorem, we have only to show that condition $3a$ is satisfied. Let $f(x)$ be a function of $L^{p_1}(0,\infty)$ and (with the usual notation) consider

$$g_a(x)-g_b(x) = \sum_{k=1}^{n} a_k \int_a^b f(y)\omega_{\lambda_k}(xy)\,dy + \int_a^b f(y)L(xy)\,dy$$
$$= \sum_{k=1}^{n} a_k\{g_b^k(x)-g_a^k(x)\}+\phi_b(x)-\phi_a(x) \quad \text{(say)}.$$

In Theorem 2, the proof of inequality (3.4) holds even when

$$-\tfrac{1}{2} \leqslant \Re(\lambda) < \tfrac{1}{2}-p_1^{-1},$$

since no use has been made of Lemma 4 at that stage of the proof. Hence we have, by (3.4) and Lemma 5,

$$\left(\int_0^\infty |g_a(x)-g_b(x)|^{p_1'}\,dx\right)^{1/p_1'} \leqslant \sum_{k=1}^{n} |a_k|\left(\int_0^\infty |g_b^k(x)-g_a^k(x)|^{p_1'}\,dx\right)^{1/p_1'}+$$
$$+\left(\int_0^\infty |\phi_b(x)-\phi_a(x)|^{p_1'}\,dx\right)^{1/p_1'}$$
$$\leqslant C_{p_1,\lambda}\left(\int_a^b |f(x)|^{p_1}\,dx\right)^{1/p_1}. \tag{4.8}$$

It follows from this that the function

$$g(x) = \underset{a\to\infty}{\mathfrak{L}_{p_1'}} \int_0^a f(y)\omega_\lambda(xy)\,dy \tag{4.9}$$

exists, and letting $a \to 0$, $b \to \infty$ in (4.8), we see that $g(x)$ satisfies the inequality (4.2). To complete the proof we have also to show that $g(x)$ is given by (4.1), and here we need the condition $4b$, which is not the same as $3b$.

If $\Re(\lambda) > 0$, we have from (4.9) in the usual way

$$\int_\delta^y x^{\lambda-\frac{1}{2}}g(x)\,dx = \int_0^\infty f(t)\,dt \int_\delta^y x^{\lambda-\frac{1}{2}}\omega_\lambda(xt)\,dx$$
$$= \int_0^\infty f(t)\left\{y^{\lambda-\frac{1}{2}}\frac{\chi_\lambda(yt)}{t} - \delta^{\lambda-\frac{1}{2}}\frac{\chi_\lambda(\delta t)}{t}\right\}dt.$$

But

$$\left|\delta^{\lambda-\frac{1}{2}}\int_0^\infty f(t)\frac{\chi_\lambda(\delta t)}{t}\,dt\right| \leqslant \delta^{\Re(\lambda)-\frac{1}{2}}\left(\int_0^\infty |f(t)|^{p_1}\,dt\right)^{1/p_1}\left(\int_0^\infty \left|\frac{\chi_\lambda(\delta t)}{t}\right|^{p_1'}dt\right)^{1/p_1'}$$
$$= \delta^{\Re(\lambda)-\frac{1}{2}+p_1^{-1}}\left(\int_0^\infty |f(t)|^{p_1}\,dt\right)^{1/p_1}\left(\int_0^\infty \left|\frac{\chi_\lambda(u)}{u}\right|^{p_1'}dt\right)^{1/p_1'},$$

and this tends to zero as $\delta \to 0$. Hence

$$\int_0^y x^{\lambda-\frac{1}{2}} g(x)\, dx = y^{\lambda-\frac{1}{2}} \int_0^\infty f(t) \frac{\chi_\lambda(yt)}{t}\, dt. \tag{4.10}$$

Similarly, when $\Re(\lambda) < \frac{1}{2} - p_1^{-1}$, we have

$$-\int_y^\infty x^{\lambda-\frac{1}{2}} g(x)\, dx = y^{\lambda-\frac{1}{2}} \int_0^\infty f(t) \frac{\chi_\lambda(yt)}{t}\, dt, \tag{4.11}$$

and (4.1) follows from (4.10) and (4.11). This completes the proof.

Example. If $\lambda = \frac{1}{2}$ and

$$\chi_\lambda(x) = -x^{\frac{1}{2}}\left\{\frac{2}{\pi} K_1(2x^{\frac{1}{2}}) + Y_1(2x^{\frac{1}{2}})\right\}, \qquad \omega_\lambda(x) = \frac{2}{\pi} K_0(2x^{\frac{1}{2}}) - Y_0(2x^{\frac{1}{2}}),$$

then conditions 1 *a* and 1 *b* are satisfied. Now

$$\omega_\lambda(x) = J_0(2x^{\frac{1}{2}}) - \sqrt{\left(\frac{2}{\pi}\right)} x^{-\frac{1}{4}} \sin(2x^{\frac{1}{2}}) + L(x),$$

where $L(x)$ is $O(-\log x)$ as $x \to 0$ and $O\{x^{-\frac{1}{4}} \exp(-2x^{\frac{1}{2}})\}$ as $x \to \infty$. Thus $L(x)$ satisfies the conditions of Lemma 5 for any $p > 1$, and in § 1 it was shown that $J_0(2x^{\frac{1}{2}})$ and $\pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \sin(2x^{\frac{1}{2}})$ are kernels satisfying the conditions 1 *a*, 2 *a*, and 2 *b* and therefore 1 *b* also, since $\lambda > 0$ in each case. Hence $\omega_\lambda(x)$ satisfies 4 *a*. Finally,

$$\frac{\chi_\lambda(x)}{x} \sim -\pi^{-\frac{1}{2}} x^{-\frac{3}{4}} \left\{\exp(-2x^{\frac{1}{2}}) + \sin\left(2x^{\frac{1}{2}} - \frac{3\pi}{4}\right)\right\} \quad \text{as} \quad x \to \infty,$$

and it is $O(-\log x)$ as $x \to 0$, so $\chi_\lambda(x)/x$ belongs to $L^r(0, \infty)$ for every r such that $\frac{4}{3} < r < \infty$. Thus the conditions of Theorem 4 are all satisfied if $p_1 = \frac{4}{3} + \delta$ $(\delta > 0)$.

5. In conclusion we shall consider the necessity of the conditions of Theorem 3. Condition 1 *a* is necessary for the existence of the transform formulae (4.1) and (1.8) when $p = 2$, and it is therefore a necessary condition for a theory of transforms for the class L^p, if p may equal 2. Condition 3 *a* is necessary if we are looking for a theory in which the χ_λ-transform of $f(x)$ satisfies an inequality of the form (4.3). Condition 3 *b* is only a sufficient condition for the convergence of the integral in (1.8). It enables us to prove (1.8) without difficulty, but for any given kernel a better condition may be found to exist.

In Theorem 3 we do not assume that $\chi_\lambda(x)$ is an integral. We shall show, however, by means of examples, that condition 1 *b* was not

introduced into Theorems 2 and 4 only to simplify the analysis, but because there are kernels $\chi_\lambda(x)$ which are not integrals, and for which we can always find functions of $L^p(0,\infty)$ (for any given $p<2$), whose χ_λ-transforms do not belong to $L^{p'}(0,\infty)$.

Consider the kernel

$$\chi_\lambda(x)=\begin{cases}1 & (x>1),\\ 0 & (x<1).\end{cases}$$

This satisfies 1 a with $\lambda=\frac{1}{2}$, and the transform formulae are

$$g(x)=\frac{1}{x}f\left(\frac{1}{x}\right),\qquad f(x)=\frac{1}{x}g\left(\frac{1}{x}\right). \tag{5.1}$$

Let
$$f(x)=\begin{cases}(1-x)^{-1/p(1+\epsilon)} & (x\leqslant 1),\\ 0 & (x>1),\end{cases} \tag{5.2}$$

where $1<p<2$ and $\epsilon>0$. This belongs to $L^p(0,\infty)$. From (5.1) and (5.2) we have

$$g(x)=\begin{cases}0 & (x<1),\\ x^{-\{1-1/p(1+\epsilon)\}}(x-1)^{-1/p(1+\epsilon)} & (x\geqslant 1),\end{cases}$$

and therefore

$$\int_0^\infty |g(x)|^{p'}\,dx=\int_1^\infty x^{-p'\{1-1/p(1+\epsilon)\}}(x-1)^{-p'/p(1+\epsilon)}\,dx.$$

This integral will not converge at the lower limit if

$$\frac{p'}{p(1+\epsilon)}\geqslant 1,$$

i.e. if
$$0<\epsilon\leqslant p'-2.$$

Since $p'>2$, there are infinitely many values of ϵ satisfying this inequality and to each there corresponds a function $f(x)$ of $L^p(0,\infty)$, whose transform does not belong to $L^{p'}(0,\infty)$.

Another discontinuous kernel for which a similar result may be proved is

$$\chi_\lambda(x)=\frac{1}{\pi}\log\left|\frac{1+x}{1-x}\right|\quad(\lambda=\tfrac{1}{2}).$$